The Operational Choi-Jamiolkowski Isomorphism[1]

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Plan

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- Leifer
- Operational formulation of the CJ Isomorphism
- Strong monogamy of correlations + Bell nonlocality = No-broadcasting/cloning
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Motivation

- What the isomorphism actually tells us: in quantum mechanics, the set of possible multipartite correlations exhibited by entangled states is equivalent to the set of possible temporal correlations exhibited by sequences of measurements on a single system over time.
- Ontic equivalence principle: phenomena which exhibit very similar statistics should be explained by the same kind of underlying mechanism.

Leifer Reformulation[2]

- Ensemble preparation: specified by a probability distribution $p(i) : i \in \{1, 2...N\}$ and a set of preparations $\{Q_i : i \in \{1, 2...N\}\}$: a procedure in which an observer draws a number i from $\{1, 2...N\}$ with probability distribution $p(i)$, and then performs the corresponding preparation Q_i.
- When the operational theory in question is quantum mechanics, every possible ensemble preparation can be described by a POVM $\{M_i\}$ and density operator ρ, where $p(i) = Tr(\rho M_i)$ and P_i is a preparation which produces the quantum state $\rho_i = \frac{\sqrt{\rho}M_i\sqrt{\rho}}{Tr(M_i\rho)}$
- For any bipartite state ρ^{ϵ}_{AB}, there exists a CPTP map ϵ and a reduced state $\rho_A = Tr_B(\rho^{\epsilon}_{AB})$ such that given any two POVMs M and O, if M^T is obtained by taking the transpose of all the measurement operators in M with respect to some fixed basis, then when we perform the ensemble preparation described by the POVM M^T and the density operator ρ_A, then evolve the state according to ϵ, then perform the measurement O, the probability that state ρ_i is prepared and then the measurement M has outcome j is the same as the joint probability of obtaining outcomes M_i and O_j when the POVM M is performed on system A and the POVM O is performed on system B for a bipartite system AB in the state ρ^{ϵ}_{AB}.
  Conversely, for any pair of a CPTP map and state ρ there exists a bipartite state ρ^{ϵ}_{AB} such that the same conditions hold, so we have defined an isomorphism between bipartite states and pairs (ρ_A, ϵ^r), where ϵ^r denotes the restriction of the CPTP map ϵ to the support of ρ_A.

[2] M. S. Leifer. “Conditional Density Operators and the Subjectivity of Quantum Operations”. In: *Foundations of Probability and Physics - 4*. Ed. by G. Adenier, C. Fuchs, and A. Y. Khrennikov. Vol. 889. American Institute of Physics Conference Series. Feb. 2007, pp. 172–186. DOI: 10.1063/1.2713456. eprint: quant-ph/quant-ph/0611233.

Operational CJ Isomorphism

- For any joint preparation $P_{123\ldots n}$ on a set of systems $S, S_2, \ldots S_n$, there exists a set of channels $T_2, T_3, \ldots T_n$ which may simultaneously be applied to the system S, such that for any set of measurements M, M_2, … M_n which may be performed on $S, S_2, \ldots S_n$, there exists an ensemble preparation P for the system S such that the distribution $p_{P_{123\ldots n};M,M_2\ldots M_n}$ is the same as the distribution $p_{P;T_2\ldots T_n;M_2\ldots M_n}$.

  Conversely, for any set of channels $T_2, T_3, \ldots T_n$ which may simultaneously be applied to the system S to produce a set of systems $S_2, \ldots S_n$, there exists a joint preparation $P_{123\ldots n}$ for systems $S, S_2, S_3 \ldots S_n$ such that for any ensemble preparation P which may be performed for system S and any set of measurements $M_2, M_3, \ldots M_n$ which may be performed on the products $S_2, \ldots S_n$, there exists a measurement M on S such that the distribution $p_{P_{123\ldots n};MM_2\ldots M_n}$ is the same as the distribution $p_{P;T_2\ldots T_n;M_2,\ldots M_n}$.

Operational CJ Isomorphism

$$
\begin{tikzcd}
M_1^1 & M_1^2 & M_1^3 & & M_2^1 & M_2^2 & M_2^3 \\
& M_1 \arrow[ul] \arrow[u] \arrow[ur] \arrow[rrrr, dash, dashed, "P_{12}"'] & & & & M_2 \arrow[ul] \arrow[u] \arrow[ur] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
M^1 & M^2 & M^3 \\
& M \arrow[ul] \arrow[u] \arrow[ur] & \\
& {} \arrow[u, "T"'] & \\
Q_1 \arrow[ur] & Q_2 \arrow[u] & Q_3 \arrow[ul] \\
1 \arrow[u] & 2 \arrow[u] & 3 \arrow[u] \\
& {} \arrow[ul, "p(1)"] \arrow[u, "p(2)"'] \arrow[ur, "p(3)"'] &
\end{tikzcd}
$$

Strong monogamy of correlations + Bell nonlocality = No-broadcasting/cloning

- No broad-casting: there is no universal broadcasting map in quantum mechanics
  - Operational state: Given two preparation procedures P_a, P_b which appear in an operational theory, these procedures produce the same *operational state* iff when a system is prepared using one of these procedures, there is no subsequent measurement or sequence of measurements which can give us any information about whether the system was prepared using P_a or P_b.
  - Operational no broad-casting: there is no map which broadcasts any set of operational states.
- Monogamy: the amount of entanglement a quantum system has with one system limits the amount of entanglement it can share with other systems
  - Measure correlations in terms of the CHSH quantity:

$$\mathscr{B}_{AB} := \langle AB\rangle_{P_{AB},M_A^0,M_B^0} + \langle AB\rangle_{P_{AB},M_A^0,M_B^1} + \langle AB\rangle_{P_{AB},M_A^1,M_B^0} - \langle AB\rangle_{P_{AB},M_A^1,M_B^1}$$

  - **Strong monogamy of correlations[3]:** An operational theory obeys strong monogamy of correlations iff for any joint preparation of three systems S_A, S_B, S_C, for any choice of measurements M_A^0, M_A^1 on S_A, any choice of measurements M_B^0, M_B^1 on S_B, and any choice of measurements M_C^0, M_C^1 on S_C, the associated CHSH quantities satisfy:

$$\mathscr{B}_{AB}(P_{ABC}, M_A^0, M_A^1, M_B^0, M_B^1)^2 + \mathscr{B}_{BC}(P_{ABC}, M_B^0, M_B^1, M_C^0, M_C^1)^2 \leq 8$$

- Bell non-locality: some CHSH quantity greater than two

[3] B. Toner and F. Verstraete. "Monogamy of Bell correlations and Tsirelson's bound". In: *eprint*

Strong monogamy of correlations + Bell nonlocality = No-broadcasting/cloning

- In an operational theory which obeys the operational Choi-Jamiolkowksi isomorphism and exhibits Bell nonlocality, the existence of a universal broadcasting map implies that the theory violates strong monogamy of correlations

No-signaling + Bell nonlocality = Preparation Contextuality

- Ontological model: a space Λ of ontic states λ, a set of probability distributions $\mu^P(\lambda)$ giving the probability that the system ends up in the state λ when we perform the preparation procedure P, a set of response functions $\xi_{M,X}(\lambda)$ giving the probability that we obtain outcome M^x when we perform measurement M on a system whose ontic state is λ, and a set of column-stochastic matrices T^O representing the way in which the ontic state is transformed when some operation O is applied to the system.
- Preparation contextuality: an operational theory is *preparation contextual* iff it is not possible to represent the theory by a valid ontological model in which every operational state is represented by a unique probability distribution over ontic states[4].
- Operational no-signalling principle: In a process involving a set of non-communicating devices $\{D_i\} : i \in \{1 \dots N\}$ such that device D_i accepts an input variable N_i and produces an output variable O_i, let J be any subset of $\{1 \dots N\}$, let O_J be the set of variables $\{O_j : j \in J\}$, let N_J be the set of variables $\{N_j : j \in J\}$; then if the inputs $\{N_i\}$ are uncorrelated, the outcomes satisfy $p(O_J|N_1, \dots, N_n,) = p(O_J|N_J)$.

[4] R. W. Spekkens. "Contextuality for preparations, transformations, and unsharp measurements". In: *Phys Rev A* 71.5, 052108 (May 2005), p. 052108. DOI: 10.1103/PhysRevA.71.052108. eprint: quant-ph/quant-ph/0406166.

No-signaling + Bell nonlocality = Preparation Contextuality

- Given an operational theory which obeys the operational Choi-Jamiołkowski isomorphism and the no-signalling principle, if the theory is preparation non-contextual, it does not exhibit Bell nonlocality.

Information Causality = Fine-grained uncertainty relations

- A sub-theory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ of an operational theory is **d-dimensional** iff d is the smallest number such that there exists a set of $d^2 - 1$ continuous parameters in $[0, 1]$ with the following properties:
    1. Specifying the values of all $d^2 - 1$ parameters for any preparation $P \in \mathcal{P}$ fully determines the probabilities $p(M^x|P)$ for every outcome M^x of every measurement M in $\mathcal{M}$.
    2. For every possible set of values of the $d^2 - 1$ parameters, there exists a preparation $P \in \mathcal{P}$ described by those parameters.
- In some subtheory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ of an operational theory, two measurements $M_1, M_2 \in \mathcal{M}$ are **orthogonal** iff given an arbitrary unknown preparation P, the set of probabilities $\{p(M_1^x|P)\}$ and $\{p(M_2^x|P)\}$ are independent.
- Information Causality[5]: if Alice and Bob pre-share a set of devices which exhibit nonlocal correlations, and Alice receives a bit string $N_0 N_1 ... N_n$ and sends Bob a classical message M of m bits, and Bob performs a measurement with some setting k and obtains outcome O, then $\sum_r I(MO : N_r|k = r) \leq m$

[5] M. Pawlowski et al. "Information causality as a physical principle". In: *Nature* 461 (Oct. 2009), pp. 1101–1104. DOI: 10.1038/nature08400. arXiv: quant-ph/0905.2292 [quant-ph].

Information Causality = Fine-grained uncertainty relations

- If an operational theory obeys the operational Choi-Jamiołkowski isomorphism and information causality, then given any subtheory $(\mathcal{P}, \mathcal{M}, \mathcal{T}, p)$ of dimension two, for any preparation $P \in \mathcal{P}$ and any pair of orthogonal measurements $M_1, M_2 \in \mathcal{M}$, and any two outcomes M_1^m, M_2^n of the measurements M_1, M_2, we must have:

$$p(M_1^m|M) + p(M_2^n|M') \leq 1 + \frac{1}{\sqrt{2}}$$

- Derivation inspired by Oppenheim/Wehner[6]

[6] J. Oppenheim and S. Wehner. "The Uncertainty Principle Determines the Nonlocality of Quantum Mechanics". In: *Science* 330 (Nov. 2010), p. 1072. DOI: 10.1126/science.1192065. arXiv: quant-ph/1004.2507 [quant-ph].

Questions?