A few puzzles in open quantum dynamics

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Celebrating the Choi-Jamiołkowski Isomorphism
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Joint ongoing work with D. Chruściński, R. Floreanini and G. Nichele

Outline I

Redfield vs Davis

Redfield vs Davis

Non-Positivity vs Complete Positivity

Redfield vs Davis

Non-Positivity vs Complete Positivity

- Total Hamiltonian: $H_T = H_S + H_B + \lambda\, H_{int}$
- System-bath unitary evolution:

$$\rho_T(t) = U_T(t)\rho_S \otimes \rho_B\, U_T^\dagger(t)\ , \quad U_T(t) = \exp(-itH_T)$$

- Open system reduced dynamics:

$$\rho_S \mapsto \rho_S(t) = \mathrm{Tr}_B \rho_T(t)$$

- Weak-coupling limit: extraction of a semigroup

$$\rho_S \mapsto \rho_S(t) = \gamma_t[\rho_S]\ , \quad \gamma_t \circ \gamma_s = \gamma_s \circ \gamma_t = \gamma_{t+s}\ ,\ 0 \leq s, t$$

Redfield Markovian approximation: γ_t NOT even Positive

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Slipped initial conditions DO NOT eliminate all problems: others emerge that are related to entanglement, through the local open dynamics

$$\boxed{\Gamma_t := \mathrm{id} \otimes \gamma_t}$$

1-qubit

F.B., D. Chruściński, R. Floreanini, OSID 29, 2022

- Hamiltonians and bath equilibrium state:

$$H_S = \omega\,\sigma_3\ ,\ H_{int} = \sum_{i=1}^{3} \sigma_i \otimes B_i\ ,\ \rho_B = \frac{\mathrm{e}^{-\beta H_B}}{Z_\beta}$$

- Diagonal bath correlation functions:

$$G_{ii}(s) = \mathrm{Tr}_B\Big[\rho_B\, B_i(s) B_i\Big] = \Big[G_{ii}(-s)\Big]^\dagger \neq 0$$

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Redfield weak-coupling limit

- Master equation: $\partial_t \rho_S(t) = \mathbb{L}[\rho_S(t)] = -i[H_S\,,\,\rho_S(t)] + \mathbb{D}[\rho_S(t)]$

$$\rho_S(t) = \gamma_t[\rho_S]\ =\ \exp\Big(t\,\mathbb{L}\Big)[\rho_S]$$

$$\mathbb{D}[\rho] = -\lambda^2 \int_0^\infty \mathrm{d}s\,\mathrm{Tr}_B\Big(\Big[H_{int}(s)\,,\,\Big[H_{int}\,,\,\rho\otimes\rho_B\Big]\Big]\Big)$$

- Interaction representation:

$$H_{int}(s) = \sum_{i=1}^{3} \sigma_i(s) \otimes B_i(s)$$

$$\sigma_i(s) = \mathrm{e}^{isH_S}\, \sigma_i\, \mathrm{e}^{-isH_S} = \sum_{j=1}^{3} U_{ij}(s)\, \sigma_j \ , \ B_i(s) = \mathrm{e}^{isH_B}\, B_i\, \mathrm{e}^{-isH_B}$$

$$U_{ij}(s) = n_i\, n_j + (\delta_{ij} - n_i\, n_j)\, \cos(2\omega s) - \varepsilon_{ijk} n_k\, \sin(2\omega s)$$

Generator in GKSL form

$$\mathbb{D}[\rho_S(t)] = -i\Big[H_{LS}\, ,\, \rho\Big] + \sum_{i,j=1}^{3} \mathcal{C}_{ij}\Big(\sigma_j \rho\, \sigma_i - \frac{1}{2}\Big\{\sigma_i \sigma_j,\, \rho\Big\}\Big)$$

- Lamb-shift correction:

$$H_{LS} = \frac{\lambda^2}{2}\,\delta\omega\,\sigma_3$$

$$\delta\omega = \int_0^\infty ds\,\sin(2\omega s)\Big(G_{11}(s)+G_{11}(-s)+G_{22}(s)+G_{22}(-s)\Big)$$

- Kossakowski matrix: $\mathcal{C} = \begin{pmatrix} \mathcal{C}_{11} & \mathcal{C}_{12} & 0 \\ \mathcal{C}_{21} & \mathcal{C}_{22} & 0 \\ 0 & 0 & \mathcal{C}_{33} \end{pmatrix}$

$$\mathcal{C}_{11} = \frac{\lambda^2}{2}\int_{-\infty}^{\infty} \mathrm{d}s\,\mathrm{e}^{2i\omega s}\Big(G_{11}(s)+G_{11}(-s)\Big)\ ,$$

$$\mathcal{C}_{22} = \frac{\lambda^2}{2}\int_{-\infty}^{\infty} \mathrm{d}s\,\mathrm{e}^{2i\omega s}\Big(G_{22}(s)+G_{22}(-s)\Big)\ ,$$

$$\mathcal{C}_{12} = \lambda^2\int_0^{\infty} \mathrm{d}s\,\sin(2\omega s)\Big(G_{22}(s)-G_{11}(-s)\Big) = \big[\mathcal{C}_{21}\big]^*\ ,$$

$$\mathcal{C}_{33} = \lambda^2\int_{-\infty}^{\infty} \mathrm{d}s\,G_{33}(s)$$

Bloch vector representation

$$\rho = \frac{1}{2}\Big(\sigma_0 + \vec{\rho}\cdot\vec{\sigma}\Big)\ ,\ \mathrm{Det}[\rho] = \frac{1}{4}\Big(1 - \sum_{j=1}^{3}\rho_j^2\Big) \geq 0\ ,$$

- the state ρ as a 4-vector $|\rho\rangle \equiv (1, \rho_1, \rho_2, \rho_3)$
- Reduced dynamics in Schrödinger-like form

$$\partial_t|\rho(t)\rangle = -2\,\mathcal{L}\,|\rho(t)\rangle\ ,\ \mathcal{L} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b+\tilde{\omega} & 0 \\ 0 & b-\tilde{\omega} & \alpha & 0 \\ w & 0 & 0 & \gamma \end{pmatrix}$$

$$\begin{array}{ll} a = \mathcal{C}_{22} + \mathcal{C}_{33}\ , & b = -\mathcal{R}e\big(\mathcal{C}_{12}\big)\ , \\ \alpha = \mathcal{C}_{11} + \mathcal{C}_{33}\ , & w = -2\,\mathcal{I}m\big(\mathcal{C}_{12}\big)\ , \\ \gamma = \mathcal{C}_{11} + \mathcal{C}_{22}\ , & \widetilde{\omega} := \omega + \frac{\lambda^2}{2}\delta\omega \end{array}$$

Tendency to equilibrium

- KMS conditions:

$$G_{ij}(t) = G_{ji}(-t-i\beta) \Longrightarrow \gamma - w = \mathrm{e}^{-2\beta\omega}\,(\gamma + w) \Longrightarrow \frac{w}{\gamma} = \frac{1-\mathrm{e}^{-2\beta\omega}}{1+\mathrm{e}^{-2\beta\omega}}$$

- Unique equilibrium state:

$$\mathbb{L}[\rho] = 0 \Longrightarrow \rho_1 = \rho_2 = 0,\ \rho_3 = -\frac{w}{\gamma} \Longrightarrow \rho = \frac{\mathrm{e}^{-\beta H_S}}{\mathrm{Tr}[\mathrm{e}^{-\beta H_S}]}$$

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Lack of positivity

- Determinants can be negative at small times:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Det}[\rho(t)]\Big|_{t=0} = 2\Big[a\rho_1^2 + \alpha\rho_2^2 + 2b\rho_1\rho_2 + \rho_3\big(w + \gamma\rho_3\big)\Big]$$

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$$\boxed{\rho_3 = 0,\ \rho_1^2 + \rho_2^2 = 1 \implies \mathrm{Det}[\rho] = 0}\ ,\ \boxed{a\,\rho_1^2 + \alpha\,\rho_2^2 + 2\,b\,\rho_1\,\rho_2 \ngeq 0}$$

An inert qubit statistically coupled to a qubit evolving with the Redfield dynamics γ_t

- Local compound reduced dynamics: $\boxed{\Gamma_t = \mathrm{id} \otimes \gamma_t}$

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- *X*-states:

$$\rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23}^* & \rho_{33} & 0 \\ \rho_{14}^* & 0 & 0 & \rho_{44} \end{pmatrix}$$

- Γ_t maps *X*-states into themselves
- Reduced density matrix of the second qubit:

$$\boxed{\rho_2(t) = \mathrm{Tr}_1\left[\rho(t)\right] \geq 0}$$

Choi-Jamiołkowski matrix $\Gamma_t[\Pi]$: γ_t CP after some time

$$\Pi = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \mapsto \Gamma_t[\Pi] = \begin{pmatrix} \rho_{11}(t) & 0 & 0 & \rho_{14}(t) \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{23}^*(t) & \rho_{33}(t) & 0 \\ \rho_{14}^*(t) & 0 & 0 & \rho_{44}(t) \end{pmatrix}$$

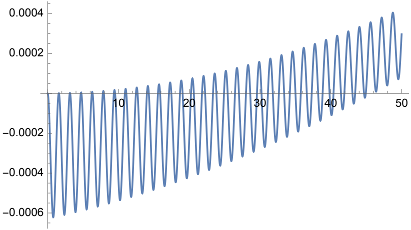

Figure: *Central matrix determinant:* $a/\omega = 0.005$, $b/\omega = 0.05$, $\alpha/\omega = 0.001$, $\gamma/\omega = 0.001$, $w = \gamma$

Entanglement of X-states via concurrence

$$\mathfrak{C}[\rho] = 2\max\Big\{0,\ |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}},\ |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}\Big\}$$

- A 4-parameter family of X-states:

$$\tilde{\rho} = \begin{pmatrix} \mu & 0 & 0 & u \\ 0 & \nu & iv & 0 \\ 0 & -iv & 1-2\mu-\nu & 0 \\ u & 0 & 0 & \mu \end{pmatrix}, \qquad \mu, \nu, u, v \in \mathbb{R}$$

$$\mu \geq 0\,, \quad \nu \geq 0\,, \quad 0 \leq 2\mu + \nu \leq 1$$

$$u^2 \leq \mu^2\,, \quad v^2 \leq \nu(1 - 2\mu - \nu)\,.$$

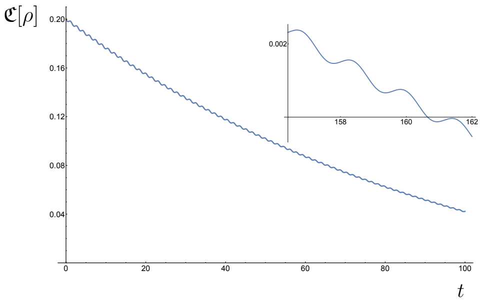

Figure: *Concurrence as a function of time in units* $1/\omega$*, with* $a/\omega = 0.005$*,* $b/\omega = 0.05$*,* $\alpha/\omega = 0.001$*,* $\gamma/\omega = 0.001$*,* $w = \gamma$*, for an initial state with* $\mu = 0.025$*,* $\nu = 0.1$*,* $u = 0.02$ *and* $v = 0.125$

Local (tensorized) reduced dynamics

$$\gamma_t \text{ CP} \Leftrightarrow \mathrm{id} \otimes \gamma_t \text{ P}$$

More physically: two open quantum systems with same local interactions with a common environment

What about $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$?

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More physically: two open quantum systems with same local interactions with a common environment

What about $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$?

- $\gamma_t \otimes \gamma_t$ positive (P) $\Leftrightarrow \gamma_t$ CP

  F.B., R. Floreanini, R. Romano, JPA 35, 2002

- $\gamma_t^{(1)} \otimes \gamma_t^{(2)}$ can be P without $\gamma_t^{(1,2)}$ being both CP

  F.B., R. Floreanini, M. Piani, OSID 11, 2004

Time-dependent generators: $\partial_t \rho_S(t) = \mathbb{L}_t[\rho_S(t)]$

$$\mathbb{L}_t[\rho_S] = -i\Big[H(t)\,,\,\rho_S(t)\Big] + \mathbb{D}_t[\rho_S(t)]$$

$$\mathbb{D}_t[\rho_S] = -i\Big[H_{LS}(t)\,,\,\rho_S\Big] + \sum_{i,j=1}^{3} \mathcal{C}_{ij}(t)\Big(\sigma_j \rho\, \sigma_i - \frac{1}{2}\Big\{\sigma_i\sigma_j,\, \rho\Big\}\Big)$$

- $\gamma_t : \rho_S \mapsto \rho_S(t)$ can be CP without $\mathcal{C}(t) = [\mathcal{C}_{ij}(t)] \geq 0$
- $\mathcal{C}(t) = [\mathcal{C}_{ij}(t)] \geq 0 \Leftrightarrow \gamma_t$ CP-divisible

A. Rivas, S. Huelga: Open Quantum Systems: an Introduction, Springer 2012

$$\boxed{\gamma_t = \gamma_{t,s} \circ \gamma_s}\ , \qquad \boxed{\gamma_{t,s} \quad CP \quad t \geq s \geq 0}$$

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$\boxed{\gamma_t = \gamma_{t,s} \circ \gamma_s}$, $\boxed{\gamma_{t,s} \quad CP \quad t \geq s \geq 0}$

$\gamma_t \otimes \gamma_t$ P-divisible $\Leftrightarrow \gamma_t$ CP-divisible

F.B., D.Chruściński, S.Filippov, PRA 95, 2017

1-qubit non-Markovianity

$$\begin{aligned}\partial_t\rho_S(t) &= \frac{1}{2}\Big(\sigma_1\rho_S(t)\sigma_1-\rho_S(t)\Big)+\frac{1}{2}\Big(\sigma_2\rho_S(t)\sigma_2-\rho_S(t)\Big)\\ &- \frac{\tanh t}{2}\Big(\sigma_3\rho_S(t)\sigma_3-\rho_S(t)\Big)\end{aligned}$$

- Pauli map:

$$\gamma_t[\sigma_1]=\mathrm{e}^{-t}\,\cosh t\,\sigma_1\ ,\ \gamma_t[\sigma_2]=\mathrm{e}^{-t}\,\cosh t\,\sigma_2\ ,\ \gamma_t[\sigma_3]=\mathrm{e}^{-2t}\,\sigma_3$$

- Intertwiner: $\gamma_{t,s}=\gamma_t\circ\gamma_s^{-1}$,

$$\gamma_{t,s}[\sigma_{1,2}]=\mathrm{e}^{-(t-s)}\,\frac{\cosh t}{\cosh s}\,\sigma_{1,2}\ ,\ \gamma_{t,s}[\sigma_3]=\mathrm{e}^{-2(t-s)}\,\sigma_3$$

- Choi-Jamiołkowski matrix:

$$\mathrm{id}\otimes\gamma_t[P]=\frac{1}{4}\begin{pmatrix}1+\mathrm{e}^{-2t} & 0 & 0 & 2\mathrm{e}^{-t}\cosh t\\ 0 & 1-\mathrm{e}^{-2t} & 0 & 0\\ 0 & 0 & 1-\mathrm{e}^{-2t} & 0\\ 2\mathrm{e}^{-t}\cosh t & 0 & 0 & 1+\mathrm{e}^{-2t}\end{pmatrix}$$

$$\left(1+\mathrm{e}^{-2t}\right)^2-4\mathrm{e}^{-2t}\cosh^2 t=0\Longrightarrow\gamma_t\ \mathrm{CP}\ \forall t\geq 0$$

- Kossakowski matrix: $\mathcal{C}_t=\frac{1}{2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -\tanh t\end{pmatrix}$
- $\gamma_t:\rho_S\mapsto\gamma_t[\rho_S]$ never CP-divisible
- $\gamma_t:\rho_S\mapsto\gamma_t[\rho_S]$ always P-divisible:

D. Chruściński, F.A. Wudarski, PLA 377, 2013.

$$\boxed{1-\tanh t\geq 1\quad\forall t\geq 0\ \Longrightarrow\ \gamma_{t,s}\ P}$$

Back-flow of information (BFI)

γ_t P-divisible, NOT CP-divisible $\Longrightarrow$ $\gamma_t \otimes \gamma_t$ NOT P-divisible

$\Downarrow$ $\Downarrow$

NO BFI BFI

Back-flow of information

from environment to system due to an open dynamics Γ_t between $t \geq s \geq 0$ if

$$\|\Gamma_t[\Delta]\|_1 \; > \; \|\Gamma_s[\Delta]\|_1 \;, \quad \|X\|_1 := \mathrm{Tr}(\sqrt{X^\dagger X})$$

for a Hellstrom matrix

$$\Delta = \mu\rho_1 - (1-\mu)\rho_2, \quad 0 \leq \mu \leq 1$$

- Γ_t P iff contractive

$$\|\Gamma_t[X]\|_1 \; \leq \; \|X\|_1 \qquad \forall X$$

- γ_t P-divisible $\Longrightarrow$ NO backflow of information:

$$\gamma_{t,s}\ P \;\Longrightarrow\; \|\gamma_t[\Delta]\|_1 = \|\gamma_{t,s}\circ\gamma_s[\Delta]\|_1 \leq \|\gamma_s[\Delta]\|_1$$

Super-activation of back-flow of information

γ_t P-divisible, NOT CP-divisible | $\gamma_t \otimes \gamma_t$ NOT P-divisible

1 qubit: no back-flow
2 qubits: back-flow under local reduced dynamics

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- Super-activation of BFI is a quantum phenomenon:
  In a commutative setting (diagonal states and observables):

$$\gamma_t \text{ P-divisible} \implies \gamma_t \otimes \gamma_t \text{ P-divisible}$$

- Super-activation of BFI requires quantum discord

- Since $\gamma_{t,s} \otimes \gamma_{t,s}$ is NOT P:

$$\left\|\gamma_{t,s} \otimes \gamma_{t,s}[\Pi]\right\|_1 > \left\|\Pi\right\|_1$$

- Suppose a Hellstrom matrix Δ exists such that $\gamma_s \otimes \gamma_s[\Delta] = \lambda\Pi$, $\lambda > 0$, then

$$\left\|\gamma_t \otimes \gamma_t[\Delta]\right\|_1 = \lambda \left\|\gamma_{t,s} \otimes \gamma_{t,s}[\Pi]\right\|_1 > \lambda \left\|\Pi\right\|_1 = \left\|\gamma_s \otimes \gamma_s[\Delta]\right\|_1$$

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Is $\Delta=\lambda^{-1}\gamma_s^{-1}\otimes\gamma_s^{-1}[\Pi]$ a Hellstrom matrix for two separable states?

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- From separable isotropic states $\rho_r:=(1-r)\frac{1}{4}+r\Pi$, $0\leq r\leq 1$,

$$\Pi=\frac{1}{r}\rho_r-\frac{1-r}{r}\frac{1}{4}\Longrightarrow\gamma_s^{-1}\otimes\gamma_s^{-1}[\Pi]=\frac{1}{r}\gamma_s^{-1}\otimes\gamma_s^{-1}[\rho_r]-\frac{1-r}{r}\frac{1}{4}$$

- For sufficiently small r, $\gamma_s^{-1}\otimes\gamma_s^{-1}[\rho_r]$ is separable by continuity.

Example

Isotropic state (Fano's form):

$$\rho_r = \frac{1}{4}\left[1_4 + r(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3)\right]$$

- Preimage:

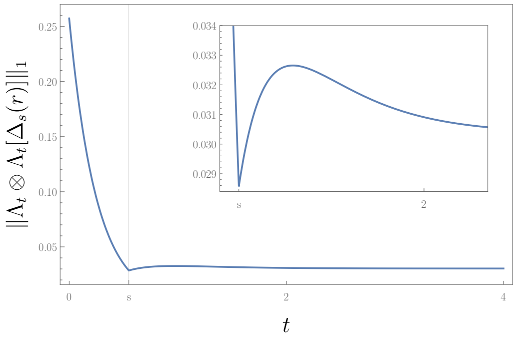
$$\begin{aligned} \widetilde{\rho}_{r,s} &:= \Lambda_s^{-1} \otimes \Lambda_s^{-1}[\rho_r] \\ &= \frac{1}{4}\left[1_4 + r\frac{\mathrm{e}^{2s}}{\cosh^2 s}(\sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2) + r\mathrm{e}^{4s}\sigma_3 \otimes \sigma_3)\right] \end{aligned}$$

- For $s = \operatorname{arctanh}(\frac{1}{2}) \approx 0.55$, $r < 1/9$ yields

$$\widetilde{\rho}_{r,s} = \frac{1}{4}\begin{pmatrix} 1+9r & 0 & 0 & \frac{9r}{2} \\ 0 & 1-9r & 0 & 0 \\ 0 & 0 & 1-9r & 0 \\ \frac{9r}{2} & 0 & 0 & 1+9r \end{pmatrix} \geq 0$$

- Separability: for $r \leq 2/27 \equiv r^*$

$$T \otimes \mathrm{id}[\rho_{r,s}] = \frac{1}{4}\begin{pmatrix} 1+9r & 0 & 0 & 0 \\ 0 & 1-9r & \frac{9r}{2} & 0 \\ 0 & \frac{9r}{2} & 1-9r & 0 \\ 0 & 0 & 0 & 1+9r \end{pmatrix} \geq 0$$



Conclusions and Outlook

- *Even if lack of positivity of Redfield reduced dynamics can be eliminated at short times by advocating an ad hoc slippage of initial conditions mechanism, problems persist at all times in the form of local generation of entanglement*
- *Super-activation of back-flow of information: when tensorized, a* 1*-qubit non-Markovian dynamics without BFI, can lead to BFI for* 2 *qubits*
- *Outlook: microscopic origin of Super-activation of BFI*: collisional models