# HEISENBERG GROUPS & DIMENSIONS OF HILBERT SPACES

- We are in a Hilbert space of fixed (finite) dimension $d$, typically because there is a group acting irreducibly on $\mathbb{C}^d$
- Maybe we are doing spectroscopy, or maybe our quantum computer is built that way
- What choices of $d$ lead to "similar" Hilbert spaces, and what choices lead to "very different" ones?

FoQaCiA

<u>Example</u> : $SU(2)$ , $SO(3) = SU(2)/\mathbb{Z}_2$

$$\mathbb{C}^\infty = \mathbb{C}^2 \oplus \mathbb{C}^3 \oplus \mathbb{C}^4 \oplus \dots$$

All finite dimensions allowed, but even and odd are "very different"

(Klein & Sommerfeld; spin superselection rule etc.)

<u>Example</u> : Heisenberg group $\mathbb{C}^\infty$

$$[Q, P] = i$$

$$U(q,p) = e^{i(pQ - qP)}$$

$$U(q_1,p_1)\,U(q_2,p_2) = \underbrace{e^{\frac{i}{2}\overbrace{(p_1 q_2 - q_1 p_2)}^{\text{symplectic form}}}}_{\text{central element}}\,U(q_1+q_2, p_1+p_2)$$

$\frac{1}{2}$

"at, or nearly at, the root of everything in the physical world"

(Eddington 1927)

Finite Weyl-Heisenberg groups $\underline{H(d)}$

$$ZX = \underbrace{\omega}_{\text{central element}} XZ \quad , \quad X^d = Z^d = \omega^d = \mathbb{1}$$

(Weyl 1932)

$\underline{\mathbb{C}^d}$:

$$Z|r\rangle = \omega^r |r\rangle$$

$$X|r\rangle = |r+1\rangle \quad , r,s \in \{0,1,\dots,d-1\} \bmod d$$

$$\omega = e^{2\pi i/d}$$

$\{X^i Z^j\}_{i,j=0}^{d-1}$ is a unitary operator basis (Schwinger 1960)

---

$d = p_1^{n_1} \cdot p_2^{n_2} \cdot \dots$

$$H(d) = H(p_1^{n_1}) \times H(p_2^{n_2}) \times \dots$$

$$\mathbb{C}^d = \mathbb{C}^{p^{n_1}} \otimes \mathbb{C}^{p^{n_2}} \otimes \dots$$

NB: $H(p^2) \neq H(p) \times H(p)$

Clifford group:

$$\{ U : U H(d) U^{-1} = H(d) \}$$

- again "splits"
- again all matrix elements are "cyclotomic", in $\mathbb{Q}(\omega_d)$
- contains a symplectic group as factor group

The even-odd distinction:

$$\{ D_{i,j} \}_{i,j=0}^{d-1} \quad : \quad D_{ij} = \underbrace{\tau^{ij}}_{\text{phase factor}} X^i Z^j$$

We want

$$D_{i,j} D_{k,l} = \underbrace{\tau^{jk-il}}_{\substack{\text{symplectic form,} \\ \text{vanishes if } [D_{ij}, D_{kl}] = 0}} D_{i+k, j+l}$$

We get

$$D_{ij} D_{kl} = \tau^{-il-jk} \omega^{jk} D_{i+k, j+l}$$

Set

$$\tau^2 = \omega$$

$$\Rightarrow \quad D_{ij} D_{k,l} = \tau^{jk-il} D_{i+k, j+l}$$

(Appleby 2005)

If $d$ is odd $\tau = \omega^{\frac{d+1}{2}}$ is a $d$th root of unity, and

$$\mathbb{Q}(\omega_d) = \mathbb{Q}(\omega_{2d})$$

If $d$ is even we have to extend the centre, and use $2d$th roots of unity

Once this extension has been made, the Clifford group contains the symplectic group

$$SL(2, \mathbb{Z}_{\bar{d}})$$

as factor group, where

$$\bar{d} = \begin{cases} d & \text{if } d \text{ is odd} \\ 2d & \text{— " — even} \end{cases}$$

Arithmetic modulo $\bar{d}$

This is an important distinction

## Wigner functions

If $d$ is odd the Clifford group contains a unique order 2 element

$$U_p : \quad U_p D_{i,j} U_p^{-1} = D_{-i,-j}$$

This is parity, and leads to an operator basis

$$A_{i,j} = D_{i,j} U_p D_{-i,-j}$$

and from there to a "nice" Wigner function.

If $d$ is even things are much less accomodating

## Contextuality

$d=2$ (even)

2dth root of unity

$D_{1,0} = X \qquad D_{0,1} = Z \qquad D_{1,1} = iXZ \equiv Y$

$$X^2 = Y^2 = Z^2 = \mathbb{1}$$

Mermin square

| | | | |
|---|---|---|---|
| $\mathbb{1}\cdot Z$ | $Z\cdot\mathbb{1}$ | $Z\cdot Z$ | $\to \mathbb{1}\cdot\mathbb{1}$ |
| $X\cdot\mathbb{1}$ | $\mathbb{1}\cdot X$ | $X\cdot X$ | $\to \mathbb{1}\cdot\mathbb{1}$ |
| $X\cdot Z$ | $Z\cdot X$ | $Y\cdot Y$ | $\to \mathbb{1}\cdot\mathbb{1}$ |
| $\downarrow$ $\mathbb{1}\cdot\mathbb{1}$ | $\downarrow$ $\mathbb{1}\cdot\mathbb{1}$ | $\downarrow$ $-\mathbb{1}\cdot\mathbb{1}$ | |

(Many possible references, including Gross; Zhu; Okay & Raussendorf, ... )

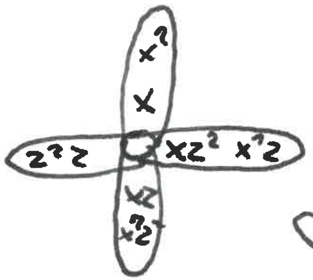
# Mutually Unbiased Bases

$$|\langle e_i | f_j \rangle|^2 = \frac{1}{d}$$

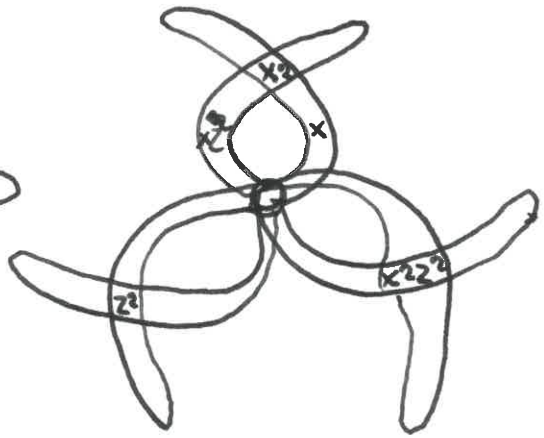
"complementarity"

## Theorem I (Bandyopadhyay et al)

A UOB that splits into $d+1$ sets of $d$ commuting operators (including $\mathbb{1}$) is equivalent to $d+1$ MUB




$H(3)$




$H(4) \neq H(2) \times H(2)$

$$\mathbb{C}P^{d-1} \curvearrowright \mathbb{R}^{d^2-1} = \mathbb{R}^{(d+1)(d-1)}$$

$(d+1)$ totally orthogonal $(d-1)$-planes

ON-basis in $\mathbb{C}^d$ spans ~~a~~ $(d-1)$-plane

$$\therefore \quad \# \, MUB \leq d+1$$

## Theorem II (Aschbacher et al)

The only groups forming such UOB are Heisenberg groups in $d = p^n$ (and the group is $H(p) \times H(p) \times \ldots$ )

- UOBs not of group type defy classification
- Numerical searches suggest that only 3 MUB exist when $d=6$

Now choose a regular simplex in $\mathbb{R}^{d^2-1}$ and inscribe in $\mathbb{C}P^{d-1}$

$$\sum_{I=1}^{N} |\psi_I\rangle\langle\psi_I| = c_1 \mathbb{1}$$

$$|\langle\psi_I|\psi_J\rangle|^2 = \begin{cases} 1 & \text{if } I=J \\ c_2 & \text{if } I\neq J \end{cases}$$

$$d \leq N \leq d^2$$

ON basis, $c_2 = 0$

SIC, $c_2 = \frac{1}{d+1}$

NOT $H(p_1)\times H(p_2)\times \ldots$

Conjecture: SICs exist, as orbits of $H(d)$, in all dimensions

(Zauner, Renes et al.; checked numerically by Scott & Grassl for $d \leq 193$)

First surprise: To construct them you need number fields that are not well understood ("Hilbert's 12th")

(Appleby, Flammia, McConnell, Yard)

There are some engineering motivations for looking at this but for now the main motivation is to change the boundary between physics and number theory.

Second surprise: Number theory connects seemingly very different dimensions.

- Cyclotomic fields $\mathbb{Q}(\omega_n)$ are the most general extensions of $\mathbb{Q}$ having abelian Galois group, and are classified by their "modulus" $n$ (as in "$n$th root of unity").

- SIC fields in dimension $d$ are abelian extensions of $\mathbb{Q}(\sqrt{D})$,
  - $D$ = square-free part of $(d+1)(d-3)$

  These are classified, and depend on a modulus. For SICs,
  - $\text{modulus} = \begin{cases} d & \text{if } d \text{ is odd} \\ 2d & \text{—"— even} \end{cases}$

- Stark conjectured existence of preferred, computable numbers in these fields, now known as "Stark units"

- SICs have been constructed from Stark units in $\approx 70$ different dimensions, including $d = 39604$

  (Kopp; Appleby, Bengtsson, Grassl, Harrison, McConnell)

How dimensions hang together: $\mathbb{Q}(\sqrt{D})$ $d > 3$

$D =$ square-free $(d+1)(d-3)$

Fix $d$, solve for $D$: $d \Rightarrow D$

$d = 4$ $D = 5$

$d = 5$ $D = 3$

$d = 6$ $D = 21$

$\vdots$

Fix $D$, solve for $d$:

$D \Rightarrow \boxed{\{d_n\}_{n=1}^{\infty}}$

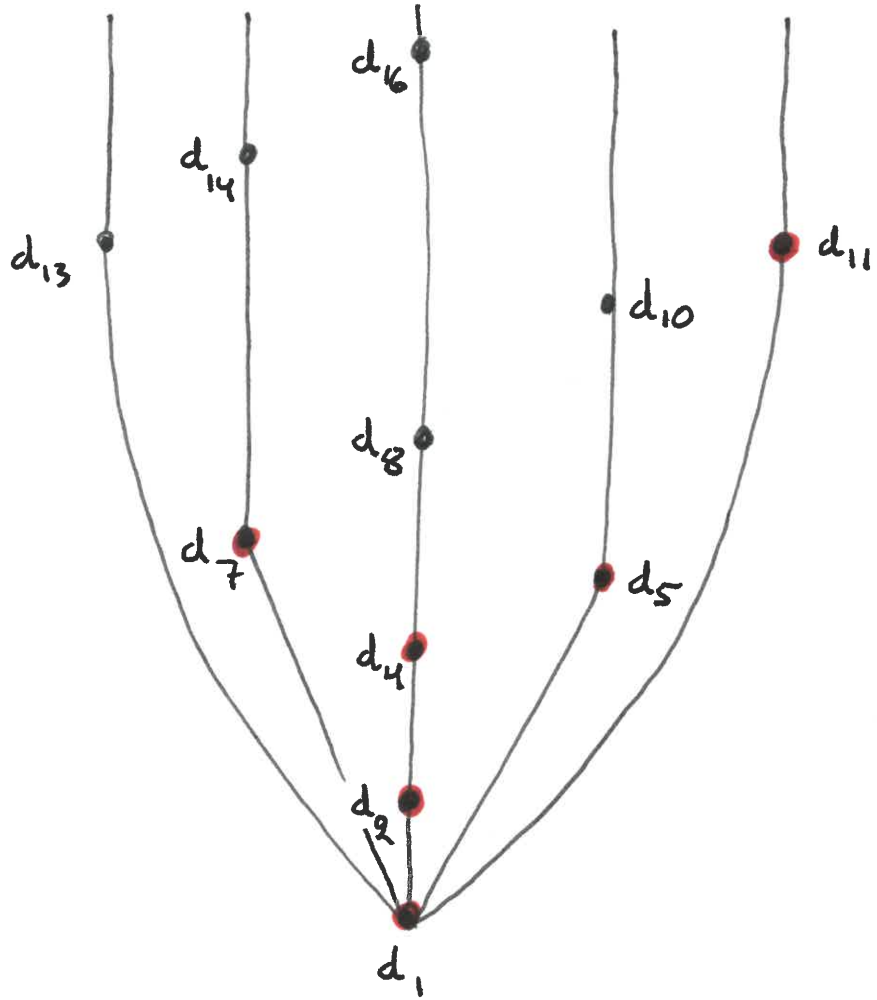
$D=5$: $\{d_n\} = \{4, 8, 19, 48, 124, 323, 844, (2208), 5779, \dots, 39604, \dots\}$

$D=3$: $\{d_n\} = \{5, 15, 195, \dots\}$ $D=21$: $\{d_n\} = \{6, 24, \dots\}$ $\dots$

- SIC symmetries grow, order $3n$
- If $\underline{d_n \mid d_m}$ connections tighten

(because the SIC field with modulus $d_n$ is a subfield of the SIC field with modulus $d_m$)

$\mathbb{Q}(\sqrt{D})$: Each sequence $\{d_n\}_{n=1}^{\infty}$ splits into lattices of tightly connected dimensions starting at $d_1$, $d_3$, $d_9$, etc




Field inclusion tower

Isomorphic towers for
$\{d_3, d_6, d_{12}, \ldots\}$, $\{d_9, d_{18} \ldots\}$

This is helpful as a rodd map, when constructing SICs

Where does it lead, in physics?

?

Meanwhile, congratulations to Choi and Jamiołkowski, who did something demonstrably important

!