A generic quantum Wielandt's inequality

Length of a matrix algebra and applications to injectivity of MPS and Kraus rank of quantum channels

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(Universität Tübingen)

**Celebrating the Choi-Jamiołkowski Isomorphism,
2 March 2023**

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A generic quantum Wielandt's inequality

Wie-generating system and Wie-length

- Consider $S \subset M_n(\mathbb{C})$.
- Assume that there is a large enough L such that

$$M_n(\mathbb{C}) = \text{span}\,\{A_1 \cdots A_L \mid A_i \in S \text{ for all } i \in [L]\}$$

Then, S is a *(Wie-)generating system* and its *Wie-length* is:

$$\mathrm{Wie}\ell(S) := \min\{L \mid M_n(\mathbb{C}) = \text{span}\,\{A_1 \cdots A_L \,;\, A_i \in S\}\}.$$

Theorem (C.-Jia '22)

$\mathrm{Wie}\ell(S) = \Theta(\log n)$ for almost all (Wie-)generating systems $S \subset M_n(\mathbb{C})$.

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Proof

Consider for simplicity $S = fA; Bg$.

Step 1

$$ABBA \mapsto (0,1,1,0) \mapsto 6 \mapsto X^6 := \begin{pmatrix} x_{11}^6 & \dots & x_{1n}^6 \\ \vdots & & \vdots \\ x_{n1}^6 & \dots & x_{nn}^6 \end{pmatrix}$$

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$$\{\dots X^i \dots X^j \dots\} \mapsto \begin{pmatrix} \dots & x_{11}^i & \dots & x_{11}^j & \dots \\ & \vdots & & \vdots & \\ \dots & x_{1n}^i & \dots & x_{1n}^j & \dots \\ \dots & x_{21}^i & \dots & x_{21}^j & \dots \\ & \vdots & & \vdots & \\ \dots & x_{nn}^i & \dots & x_{nn}^j & \dots \end{pmatrix} =: W$$

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If $\det(W) \neq 0 \Rightarrow \{\dots X^i \dots X^j \dots\}$ are l.i.

Step 3 (more detail)

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

$$\Rightarrow \det(W) = \sum_{p,q,r,s} (-1)^{f(p,q,r,s)} a_{i_1 j_1} \cdots a_{i_p j_q} b_{k_1 l_1} \cdots b_{k_r l_s} =: P(a_{ij}, b_{kl})$$

with $f(p,q,r,s) = 0$ or 1

If $\det(W) \neq 0$ then $P \not\equiv 0 \Rightarrow \{a_{ij}, b_{kl} \, : \, P(a_{ij}, b_{kl}) = 0\}$ has measure 0

$\Rightarrow \text{span}\{\dots X^i \dots X^j \dots\} = M_n(\mathbb{C})$ almost surely

Proof. Step 0

- First, consider n^2 words of length ℓ in A and B, namely products of the form

$$\underbrace{ABBAB\ldots BA}_{\ell \text{ elements}}.$$

- By some counting argument, it is clear that $\ell = \;(\log n)$.

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  or more generally

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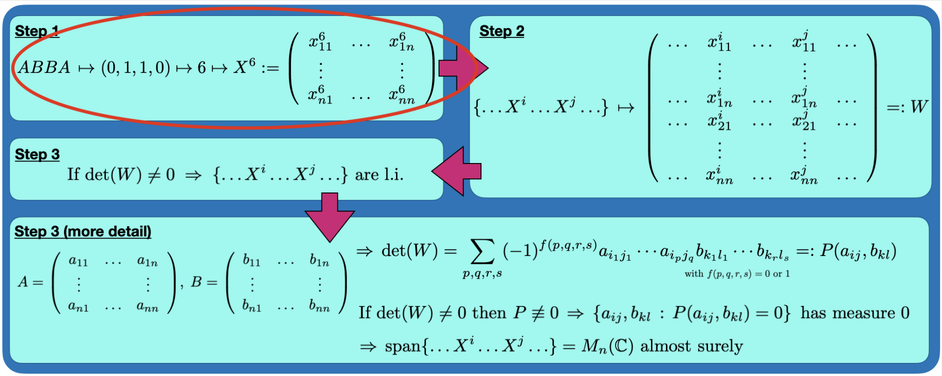

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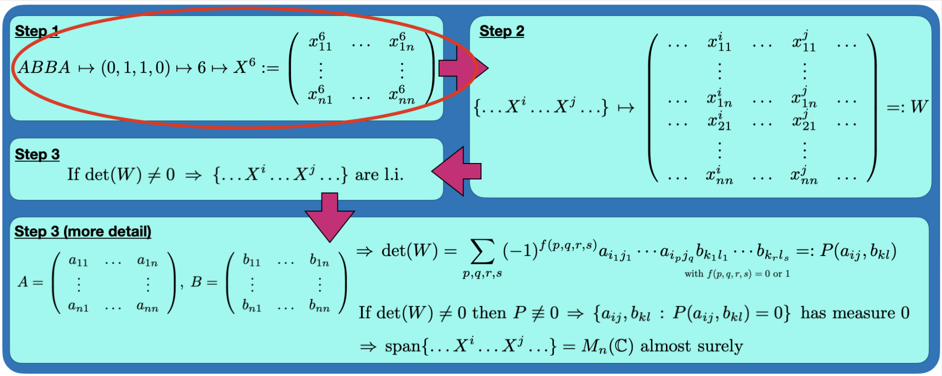

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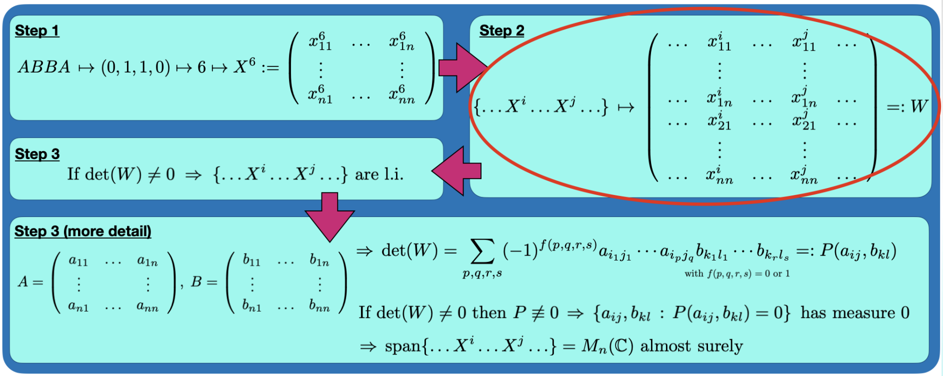

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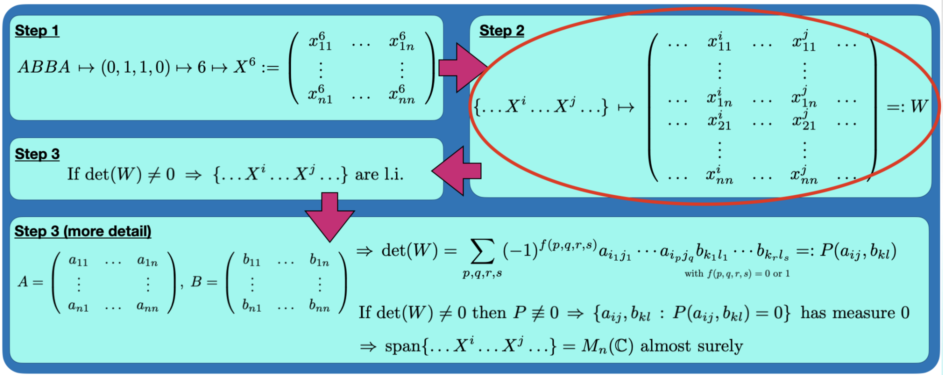
PROOF. STEP 2: VECTORIZE WORDS AND JOIN THEM IN A MATRIX.



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$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

$$\Rightarrow \det(W) = \sum_{p,q,r,s} (-1)^{f(p,q,r,s)} a_{i_1 j_1} \cdots a_{i_p j_q} b_{k_1 l_1} \cdots b_{k_r l_s} =: P(a_{ij}, b_{kl})$$

with $f(p,q,r,s) = 0$ or 1

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- The remaining part to conclude is to justify the existence of the words of Step 0.

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There are n^2 words of length $2d\log_g ne$ such that P is not the identically-zero polynomial.

Proof. Step 4: Existence of the words of Step 0.

Step 1

$$ABBA \mapsto (0,1,1,0) \mapsto 6 \mapsto X^6 := \begin{pmatrix} x_{11}^6 & \dots & x_{1n}^6 \\ \vdots & & \vdots \\ x_{n1}^6 & \dots & x_{nn}^6 \end{pmatrix}$$

Step 2

$$\{\dots X^i \dots X^j \dots\} \mapsto \begin{pmatrix} \dots & x_{11}^i & \dots & x_{11}^j & \dots \\ & \vdots & & \vdots & \\ \dots & x_{1n}^i & \dots & x_{1n}^j & \dots \\ \dots & x_{21}^i & \dots & x_{21}^j & \dots \\ & \vdots & & \vdots & \\ \dots & x_{nn}^i & \dots & x_{nn}^j & \dots \end{pmatrix} =: W$$

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Application: Kraus rank of quantum channels

Consider a quantum channel E, i.e. a completely positive trace-preserving linear map,

$$E : B(H) \to B(K) \text{ CPTP } .$$

Primitive quantum channel and index of primitivity

- The channel is *primitive* if there is an integer $\ell \in \mathbb{N}$ such that, for any positive semi-de nite matrix , the ℓ-fold application of the channel to is positive de nite, namely if

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- Using the **Choi-Jamiolkowski Isomorphism**, we construct the matrix $\omega(E) = (\mathrm{id} \otimes E)(\psi)$ with $\psi = \sum_{i,j=1}^{n} |ii\rangle\langle jj|$.
- Then, the rank of $\omega(E)$ is the *Kraus rank* of the channel.

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Moreover, the Kraus rank is lower bounded by $q(E)$.

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- The notion of full Kraus rank for a quantum channel is equivalent to that of Wie-generating system for its Kraus operators.
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$$\mathcal{E}(X) = \sum_{i=1}^{g} A_i X A_i^{\dagger},$$

then having full Kraus rank is equivalent to

$$\operatorname{span}\{X_1 \cdots X_m \,|\, X_i = A_j \text{ for } i \in [m], j \in [g]\} = \mathcal{M}_n(\mathbb{C})$$

for a minimal $\ell \in \mathbb{N}$, or, equivalently, $\mathrm{Wie}(\{A_1, \ldots, A_g\}) = \ell$.

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Corollary (C.-Jia '22)

Given a generic quantum channel $\mathcal{E} : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with Kraus operators $\{A_1, \ldots, A_g\}$, its Kraus rank (and thus its index of primitivity $q(\mathcal{E})$) is of order $\ (\log n)$.

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Application: Matrix Product States

Matrix Product State

Consider a pure quantum state $|\psi\rangle \in (\mathbb{C}^{g})^{\otimes L}$ modelling a system of L sites, each of which corresponds to a g-dimensional Hilbert space. If a translation-invariant pure state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_{i_1,\dots,i_L=1}^{g} \operatorname{tr}\left(A_{i_1}\dots A_{i_L}\right) |i_1\dots i_L\rangle$$

we say that $|\psi\rangle$ is a *Matrix Product State* (MPS) with periodic boundary conditions.

For any $L \in \mathbb{N}$, let us consider the map $\Gamma_L : M_n(\mathbb{C}) \to (\mathbb{C}^{g})^{\otimes L}$ given by

$$\Gamma_L : X \mapsto \sum_{i_1,\dots,i_L=1}^{g} \operatorname{tr}\left(X A_{i_1}\dots A_{i_L}\right) |i_1\dots i_L\rangle \, .$$

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Theorem (Pérez-García et al. '06)

Γ_L is injective if, and only if,

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- Given $L \in \mathbb{N}$ such that
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Lie algebra

Lie-generating system and Lie-length

Consider a Lie algebra $(\mathcal{A}, [\cdot,\cdot])$ and a generating system $\mathcal{U} \subset \mathcal{A}$. We define the *Lie-length* of a Lie-generating system $\mathcal{U}$ as:

$$\text{Lie}\ell(\mathcal{U}) = \min\{\ell \mid \mathcal{A} = \text{span}\{\mathcal{U}_n, n \leq \ell\}\}, \quad \text{with} \quad \mathcal{U}_1 = \text{span}\{\mathcal{U}\}, \ \mathcal{U}_n = \text{span}[\mathcal{U}_{n-1}, \mathcal{U}], \ n \geq 2.$$

As $\{\mathcal{U}_n\}$ is a grading of the Lie algebra and basis elements can thus be restricted to right-nested brackets, we could search for a basis with minimal length through a tree structure algorithm.

Lie-Tree algorithm

- At each step, the length increases by one and we compute a new set of right-nested commutators.
- We consider one of them, evaluate it as a matrix and discard it if it is linearly dependent of the previous matrices.
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Testing the "Lie-Tree" algorithm for random pairs in

su(n) for n 20, we observe that the Lie-length scales as (log n) and it does not change when we randomly choose another initial pair. Similar numerical results with the same asymptotic behaviour hold for $gl(n; \mathbb{R})$, $gl(n; \mathbb{C})$, o(n), u(n), so(n).

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Let S be a random Lie-generating set of su(n), then

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Conclusion

What is the minimum $\ell \in \mathbb{N}$ such that all words on S of length at most ℓ span $M_n(\mathbb{C})$?

$$\operatorname{span} S^{\ell} = M_n(\mathbb{C}) .$$

For any generating system S, the conjecture is $O(n)$, but the best bound is $\boxed{O(n \log n)}$.

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