A generic quantum Wielandt's inequality

Length of a matrix algebra and applications to injectivity of MPS and Kraus rank of quantum channels

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Length of a matrix algebra. Paz's conjecture

- Consider any two n-dimensional complex matrices $A, B \in M_n(\mathbb{C})$, $S := \{A, B\}$.
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$$S^{\leq \ell} := \{A, B, AA, AB, BA, BB, \ldots, \underbrace{A\ldots A}_{\ell \text{ elements}}, \underbrace{AAB\ldots BA}_{\ell \text{ elements}}, \ldots, \underbrace{BBA\ldots AB}_{\ell \text{ elements}}, \ldots, \underbrace{B\ldots B}_{\ell \text{ elements}}\}$$

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Generic quantum Wielandt's inequality (C.-Jia '22)

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A generic quantum Wielandt's inequality

Wie-generating system and Wie-length

- Consider $S \subset M_n(\mathbb{C})$.
- Assume that there is a large enough L such that

$$M_n(\mathbb{C}) = \text{span}\,\{A_1 \dots A_L \mid A_i \in S \text{ for all } i \in [L]\}$$

Then, S is a *(Wie-)generating system* and its *Wie-length* is:

$$\texttt{Wie}\ell(S) := \min\{L | M_n(\mathbb{C}) = \text{span}\,\{A_1 \dots A_L\,, A_i \in S\}\}.$$

Theorem (C.-Jia '22)

$\texttt{Wie}\ell(S) = \Theta(\log n)$ for almost all (Wie-)generating systems $S \subset M_n(\mathbb{C})$.

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Proof

Consider for simplicity $S = \{A, B\}$.

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$$ABBA \mapsto (0,1,1,0) \mapsto 6 \mapsto X^6 := \begin{pmatrix} x_{11}^6 & \dots & x_{1n}^6 \\ \vdots & & \vdots \\ x_{n1}^6 & \dots & x_{nn}^6 \end{pmatrix}$$

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$$\{\dots X^i \dots X^j \dots\} \mapsto \begin{pmatrix} \dots & x_{11}^i & \dots & x_{11}^j & \dots \\ & \vdots & & \vdots & \\ \dots & x_{1n}^i & \dots & x_{1n}^j & \dots \\ \dots & x_{21}^i & \dots & x_{21}^j & \dots \\ & \vdots & & \vdots & \\ \dots & x_{nn}^i & \dots & x_{nn}^j & \dots \end{pmatrix} =: W$$

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If $\det(W) \neq 0 \Rightarrow \{\dots X^i \dots X^j \dots\}$ are l.i.

Step 3 (more detail)

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}, \; B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

$$\Rightarrow \det(W) = \sum_{p,q,r,s} (-1)^{f(p,q,r,s)} a_{i_1 j_1} \cdots a_{i_p j_q} b_{k_1 l_1} \cdots b_{k_r l_s} =: P(a_{ij}, b_{kl})$$

with $f(p,q,r,s) = 0$ or 1

If $\det(W) \neq 0$ then $P \not\equiv 0 \Rightarrow \{a_{ij}, b_{kl} \, : \, P(a_{ij}, b_{kl}) = 0\}$ has measure 0

$\Rightarrow \text{span}\{\dots X^i \dots X^j \dots\} = M_n(\mathbb{C})$ almost surely

Proof. Step 0

- First, consider n^2 words of length ℓ in A and B, namely products of the form

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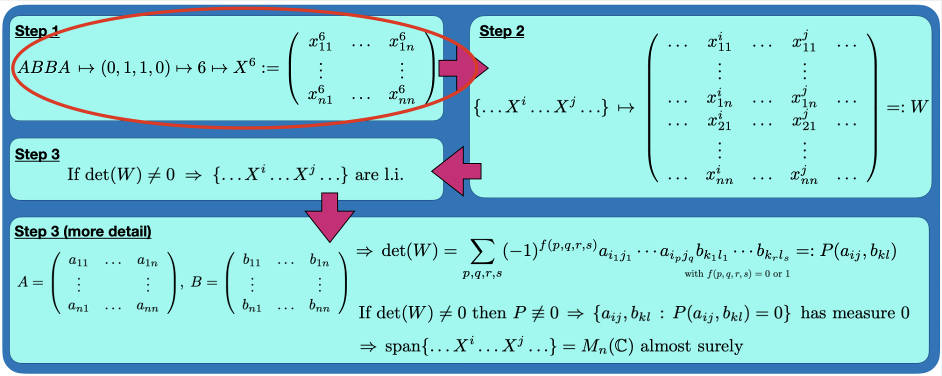

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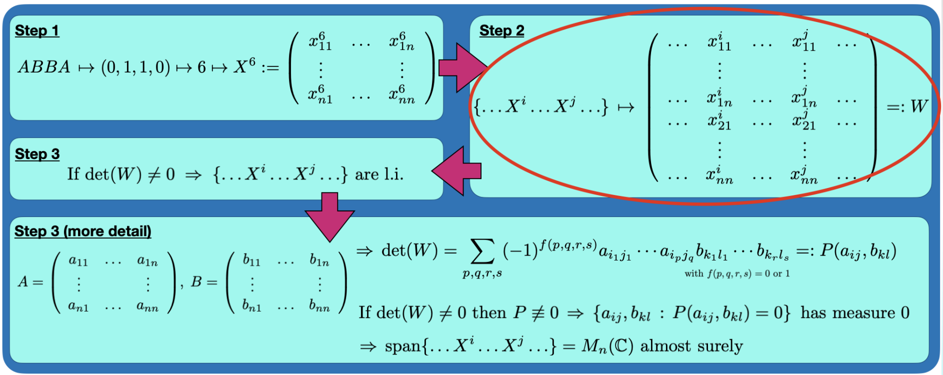

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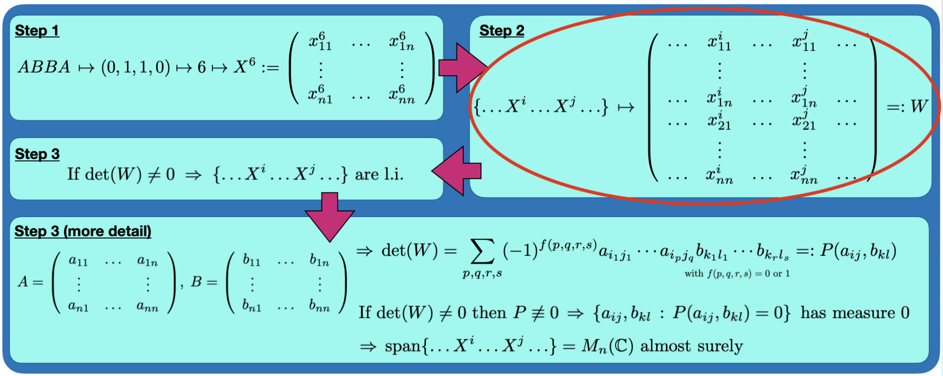

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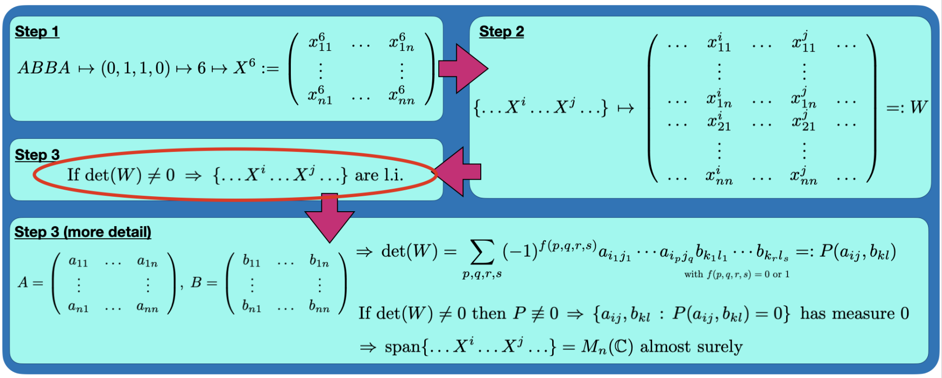

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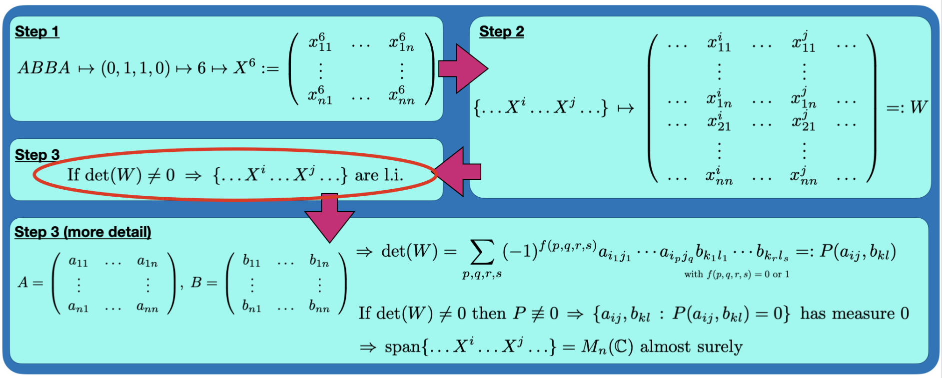

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If $\det(W) \neq 0 \Rightarrow \{\dots X^i \dots X^j \dots\}$ are l.i.

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$\Rightarrow \operatorname{span}\{\dots X^i \dots X^j \dots\} = M_n(\mathbb{C})$ almost surely

- More specifically, $\det(W)$ is actually a polynomial of $2n^2$ variables, namely $\{a_{ij}\}_{i,j=1}^n$ and $\{b_{kl}\}_{k,l=1}^n$, the coefficients of A and B respectively.
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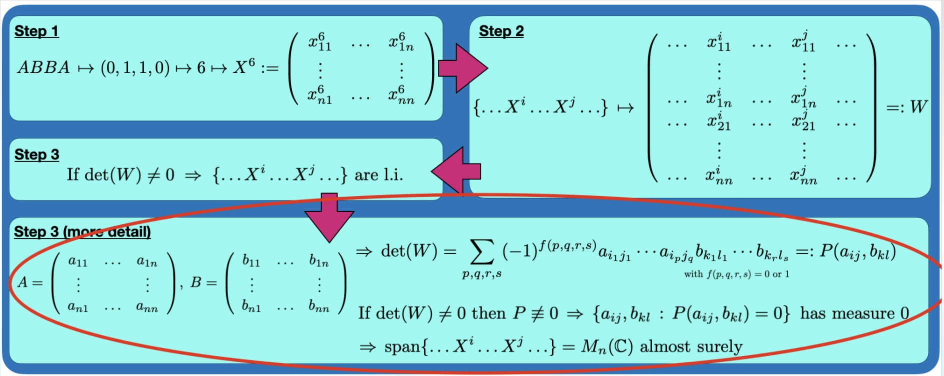

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Consider a quantum channel $\mathcal{E}$, i.e. a completely positive trace-preserving linear map,

$$\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \text{ CPTP} .$$

Primitive quantum channel and index of primitivity

- The channel is *primitive* if there is an integer $\ell \in \mathbb{N}$ such that, for any positive semi-definite matrix ρ, the ℓ-fold application of the channel to ρ is positive definite, namely if

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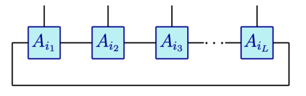
Application: Matrix Product States

Matrix Product State

Consider a pure quantum state $|\psi\rangle \in \mathbb{C}^{\otimes g^L}$ modelling a system of L sites, each of which corresponds to a g-dimensional Hilbert space. If a translation-invariant pure state $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_{i_1,\ldots,i_L=1}^{g} \operatorname{tr}\left[A_{i_1}\ldots A_{i_L}\right]|i_1\ldots i_L\rangle$$



we say that $|\psi\rangle$ is a *Matrix Product State* (MPS) with periodic boundary conditions.

For any $L \in \mathbb{N}$, let us consider the map $\Gamma_L : M_n(\mathbb{C}) \to \mathbb{C}^{\otimes g^L}$ given by

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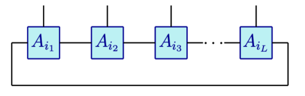

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Γ_L is injective if, and only if,

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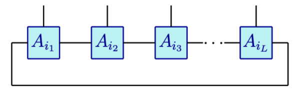

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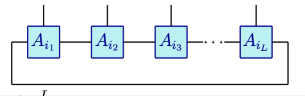

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the map Γ_L is injective with probability 1.
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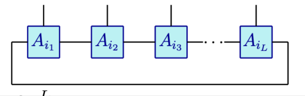

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Lie algebra

Lie-generating system and Lie-length

Consider a Lie algebra $(\mathcal{A}, [\cdot,\cdot])$ and a generating system $U \subset \mathcal{A}$. We define the *Lie-length* of a Lie-generating system U as:

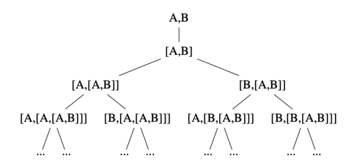
$$\texttt{Lie}\ell(U) = \min\{\ell | \mathcal{A} = \text{span}\{U_n, n \leq \ell\}\}, \quad \text{with} \quad U_1 = \text{span}\{U\};\, U_n = \text{span}[U_{n-1}, U]\,,\, n \geq 2.$$

As $\{U_n\}$ is a grading of the Lie algebra and basis elements can thus be restricted to right-nested brackets, we could search for a basis with minimal length through a tree structure algorithm.

Lie-Tree algorithm

- At each step, the length increases by one and we compute a new set of right-nested commutators.
- We consider one of them, evaluate it as a matrix and discard it if it is linearly dependent of the previous matrices.
- We repeat this with all the new right-nested commutators.

The algorithm stops when there are enough basis elements or the length reaches the dimension.



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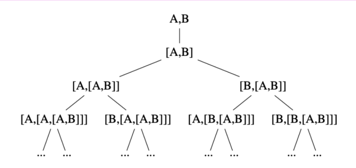

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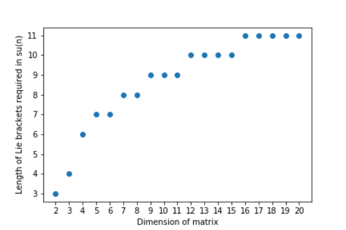
Testing the "Lie-Tree" algorithm for random pairs in $\mathfrak{su}(n)$ for $n \leq 20$, we observe that the Lie-length scales as $\Theta(\log n)$ and it does not change when we randomly choose another initial pair. Similar numerical results with the same asymptotic behaviour hold for $gl(n, \mathbb{R})$, $gl(n, \mathbb{C})$, $\mathfrak{o}(n)$, $\mathfrak{u}(n)$, $\mathfrak{so}(n)$.



Conjecture (C.-Jia '22)

Let S be a random Lie-generating set of $\mathfrak{su}(n)$, then

$$\texttt{Lie}\ell(S) = \Theta(\log n) \text{ almost surely.}$$

Lie algebra

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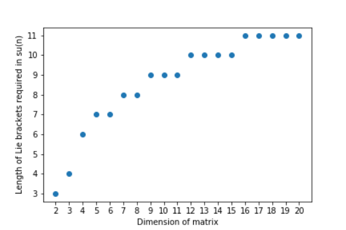

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Conclusion

What is the minimum $\ell \in \mathbb{N}$ such that all words on S of length at most ℓ span $M_n(\mathbb{C})$?

$$\operatorname{span} S^{\leq \ell} = M_n(\mathbb{C}) \, .$$

For any generating system S, the conjecture is $O(n)$, but the best bound is $\boxed{O(n \log n)}$.

What is the minimum $\ell \in \mathbb{N}$ such that all words on S of length exactly ℓ span $M_n(\mathbb{C})$?

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