HIGHER ORDER QUANTUM PROCESSES AND QUANTUM CAUSAL STRUCTURES

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SAPIENTIA ET VIRTUS

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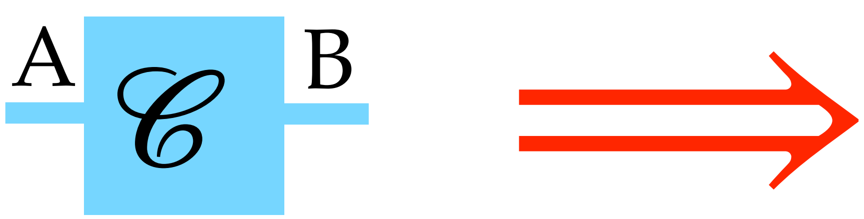
INTRODUCTION (INFORMAL)

INSPIRATION FROM CHOI-JAMIOŁKOWSKI

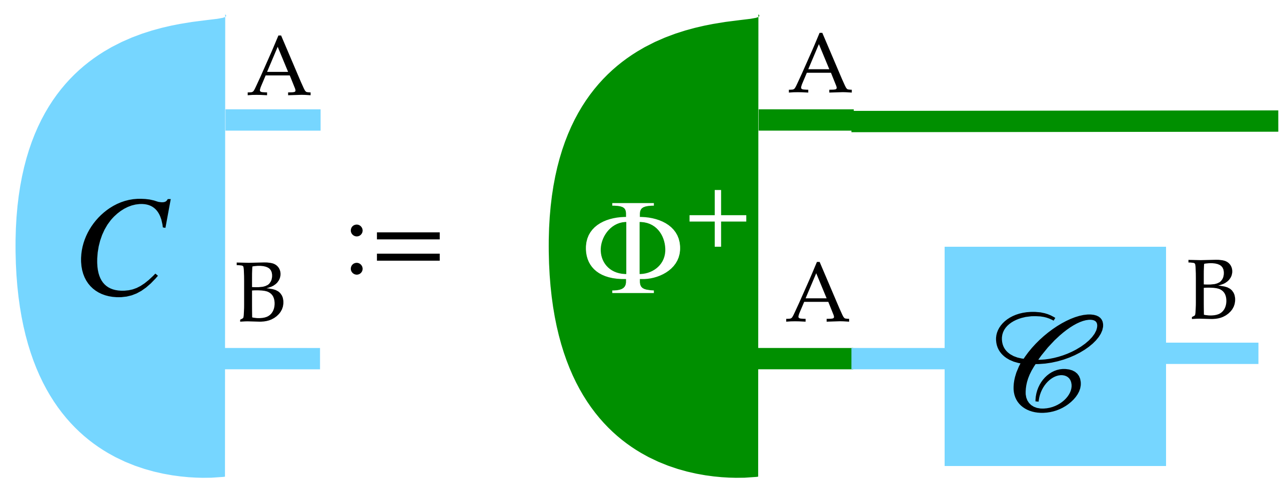
A lesson of the Choi–Jamiołkowski isomorphism: quantum states and quantum processes can be treated on the same footing.

Quantum process
(completely positive
trace non-increasing map)



isomorphism
between the cone of
unnormalized states
and the cone of
unnormalized processes

Quantum state
(positive semidefinite operator
with trace bounded by 1)



$$\Phi^+ = \frac{1}{d_A} \sum_{m,n} |m\rangle\langle n| \otimes |m\rangle\langle n|$$

QUANTUM SUPER-PROCESSES

The isomorphism between the (cone of) quantum states and the (cone of) quantum processes suggests an idea of *"quantum super-process"*.

Informally:

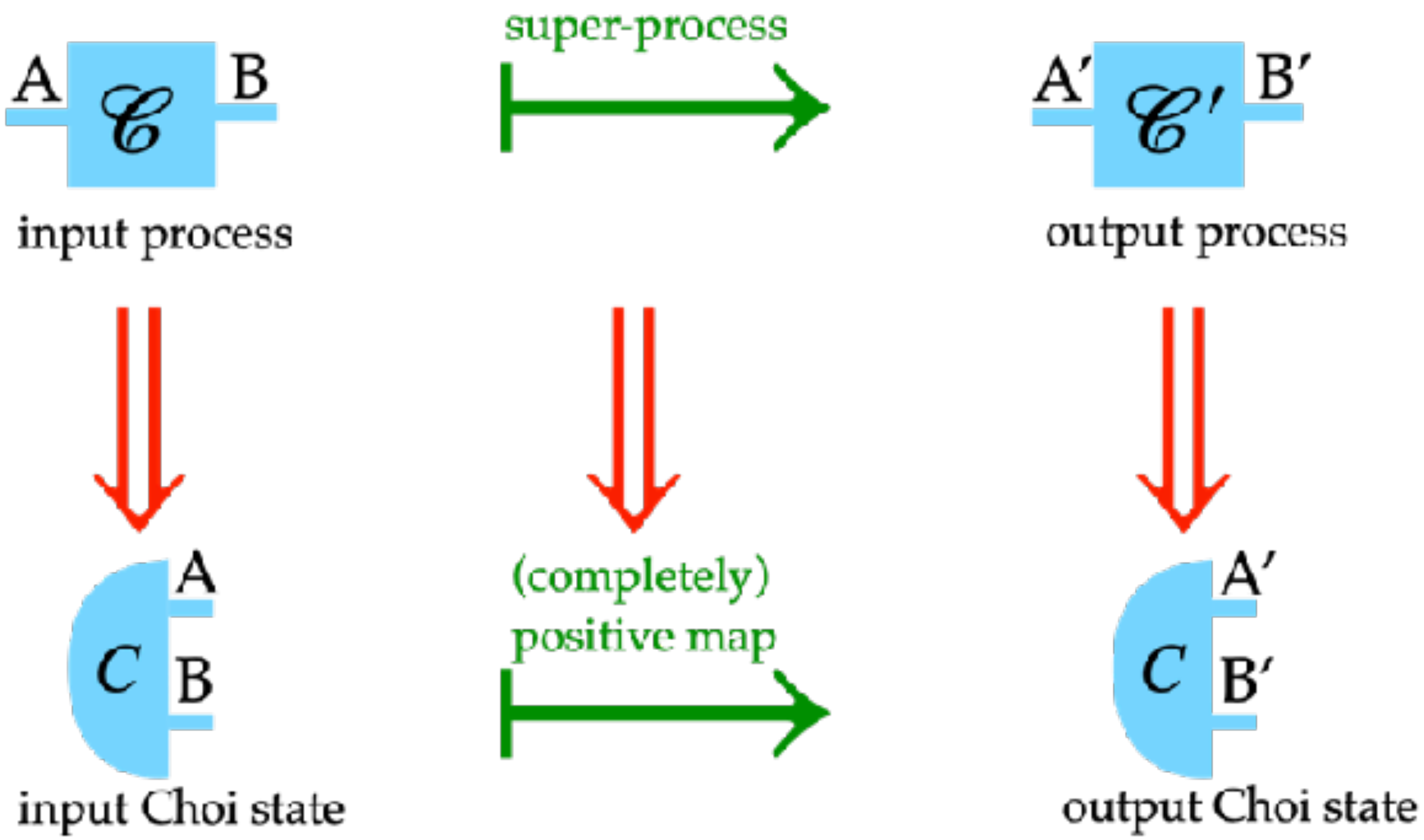
a quantum super-process should transform quantum processes
into quantum processes,

in a similar way as

a quantum process transforms quantum states
into quantum states

TRIVIAL CHARACTERIZATION?

From the Choi–Jamiołkowski isomorphism,
it is clear that a super-process can be represented by a (completely) positive map.



Chiribella, D'Ariano, and Perinotti, EPL 83, 30004 (2008);
Zyczkowski, Phys. A 41, 355302-23 (2008).

But is that all? Aren't there more conditions that a super-process should satisfy?

NORMALIZATION

No matter how much we love the Choi–Jamiołkowski isomorphism, we should't be carried away:
there still exist a difference between states and processes!

The difference is in the **normalization**:

-*normalized (a.k.a. deterministic)* states = trace-1 operators

-*normalized (a.k.a. deterministic)* processes = trace-preserving maps

The Choi–Jamiołkowski isomorphism ***not*** an isomorphism between the set of normalized states and the set of normalized processes.

Hence, super-processes are not trivially characterized as ordinary processes acting on the Choi states:
they have a different normalization condition!

PLAN FOR THE (REST OF THE) TALK

- Framework of quantum supermaps

- Higher-order quantum processes and quantum causal structures

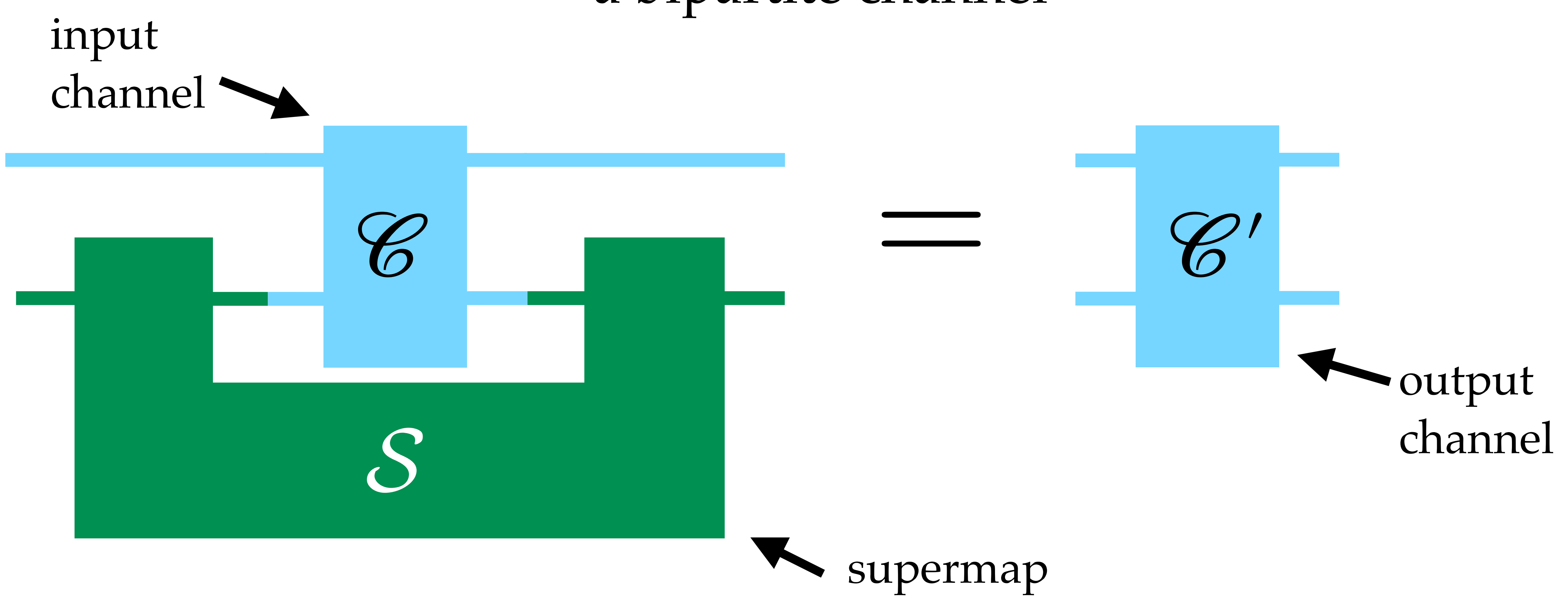
- General supermaps on subsets of quantum processes

QUANTUM SUPERMAPS

TRANSFORMATIONS OF QUANTUM CHANNELS

Deterministic supermap **[Chiribella, D'Ariano, and Perinotti, EPL 83, 30004 (2008)]**

must be **linear** (in the input CP map) and **send quantum channels (deterministic quantum processes) into quantum channels,** even when acting locally on one part of a bipartite channel



MATHEMATICAL CHARACTERIZATION

The Choi–Jamiołkowski isomorphism offers a first characterization of the deterministic supermaps:

in the Choi representation, a supermap $\mathcal{S}$ induces a completely positive map $\widehat{\mathcal{S}} : L\left(\mathbb{C}^{d_A} \otimes \mathbb{C}^{d_B}\right) \rightarrow L\left(\mathbb{C}^{d_{A'}} \otimes \mathbb{C}^{d_{B'}}\right)$
The supermap is deterministic if and only if the condition

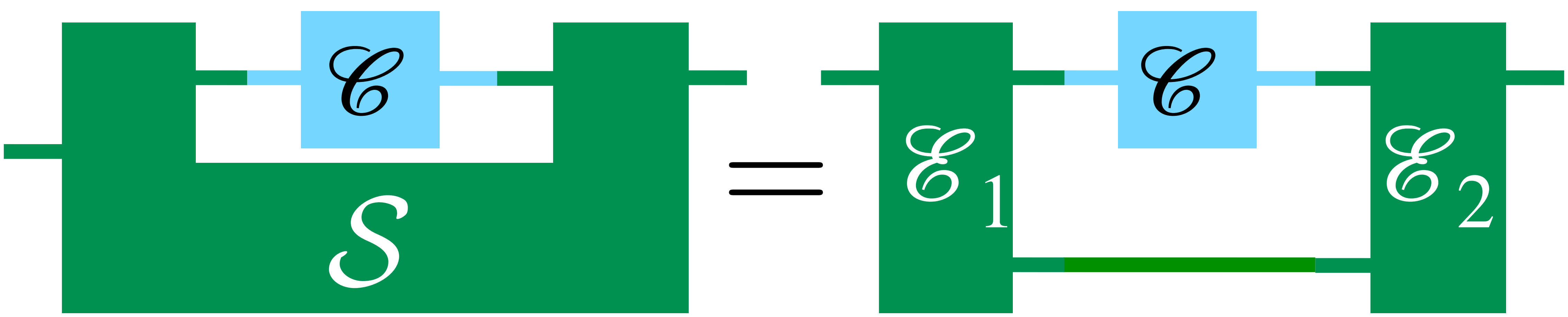
$$\operatorname{Tr}_{B'}\left[\widehat{\mathcal{S}}(C)\right] = \frac{I_{A'}}{d_{A'}} \qquad \forall C : \operatorname{Tr}_B[C] = \frac{I_A}{d_A}$$

is satisfied.

CHARACTERIZATION

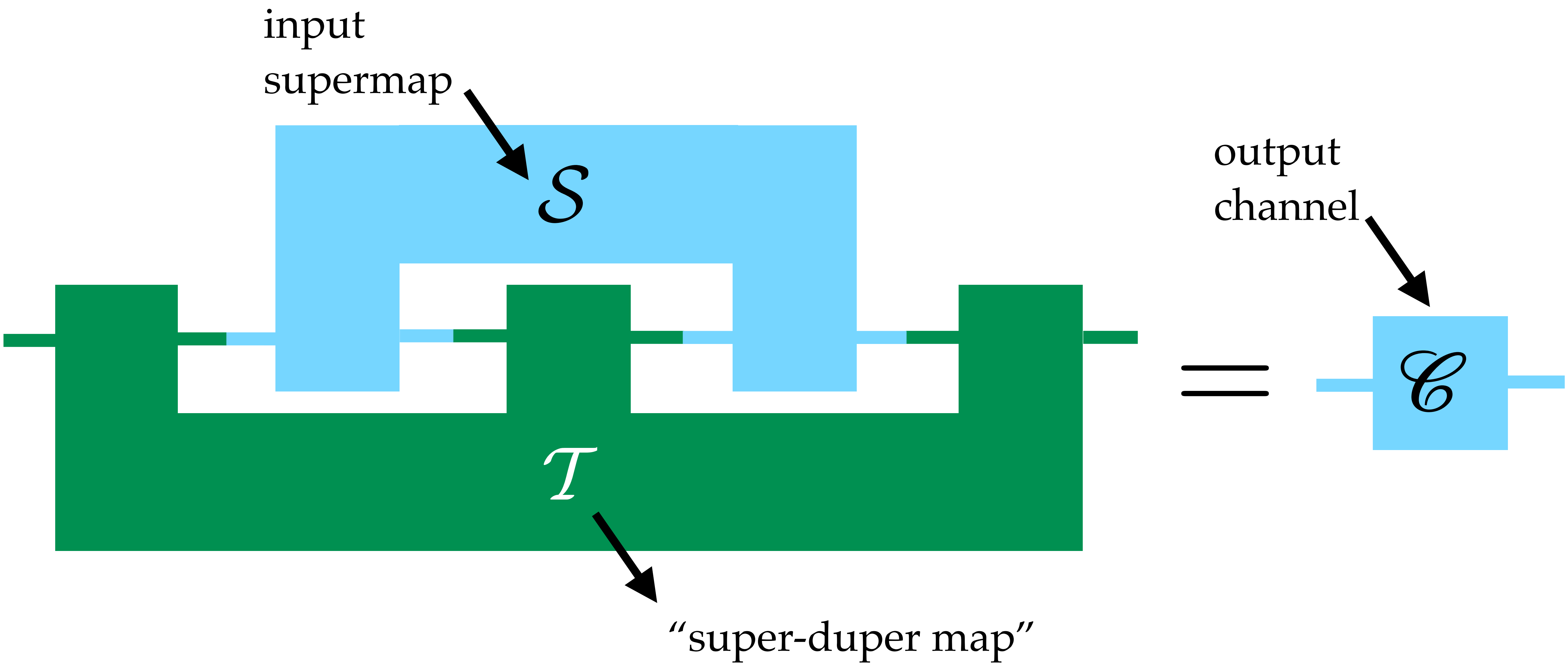
Theorem **[Chiribella, D'Ariano, and Perinotti, EPL 83, 30004 (2008)]**

Every deterministic supermap can be realized by a network of quantum channels with memory:



SUPERMAPS
OF
HIGHER ORDER

NEXT NEXT LEVEL:
MAPPING SUPERMAPS INTO CHANNELS

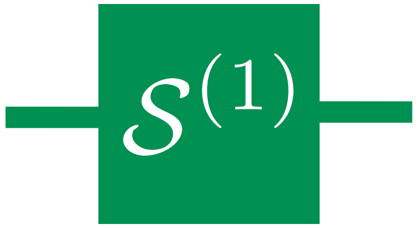


Deterministic “super-duper map”:

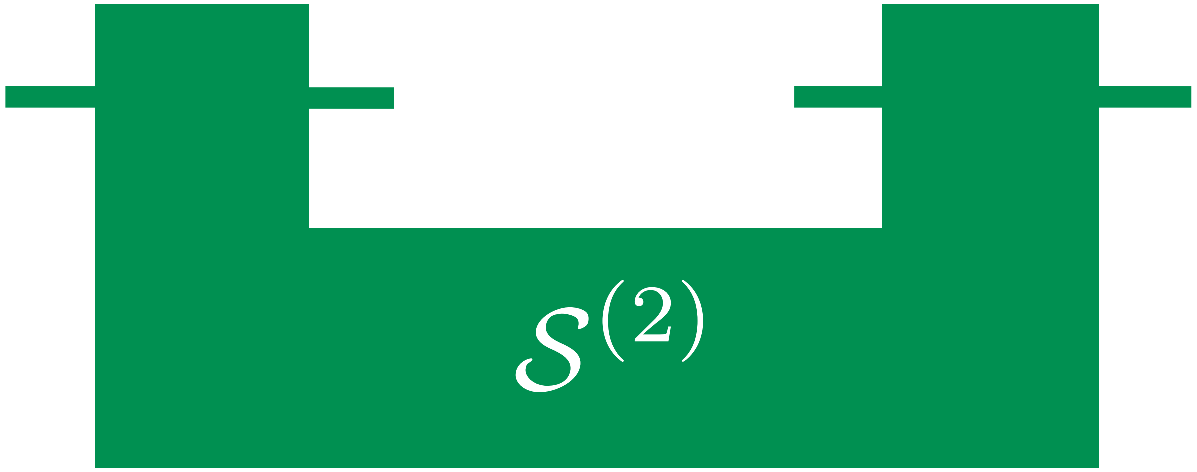
must be **linear** and **send deterministic supermaps into channels**, even when acting locally on one part of a bipartite supermap

N-MAPS

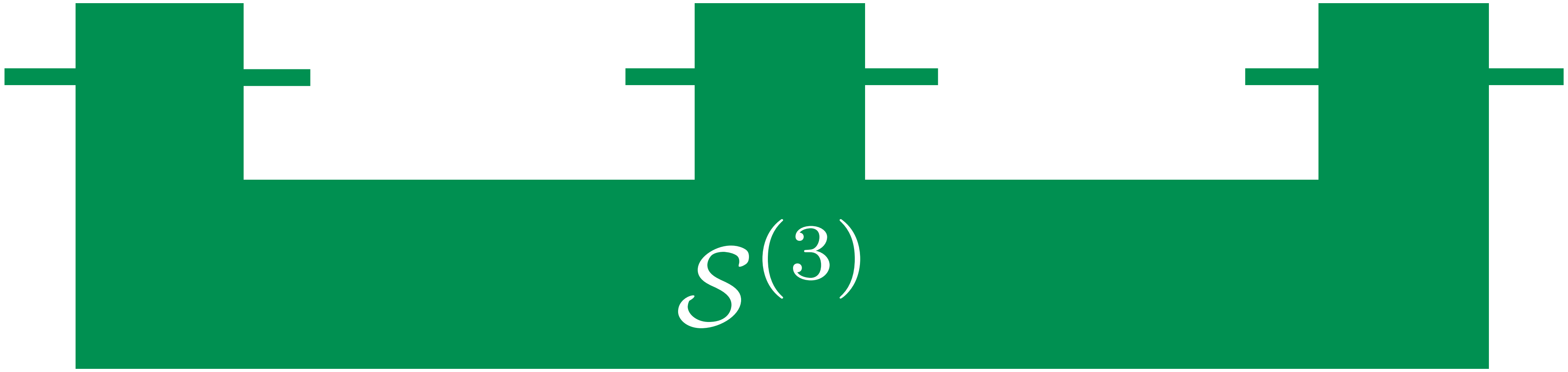
N=1 quantum channel



N=2

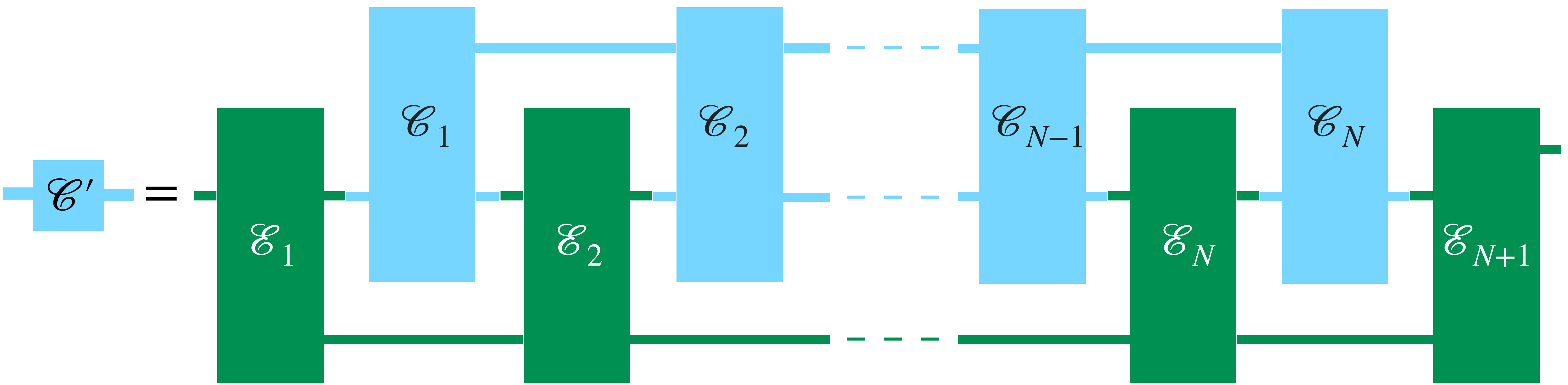


N=3



REALIZATION OF THE DETERMINISTIC N-MAPS

Theorem [GC, D'Ariano, and Perinotti, PRA A 80, 022339 (2009)]
Any deterministic N-map can be realized by a
sequential network of quantum channels with memory.

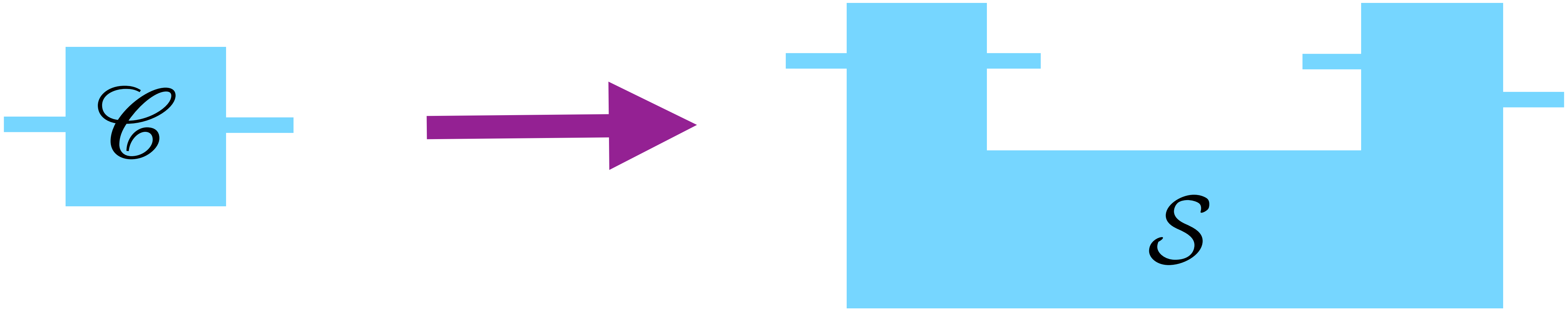


These supermaps are called "quantum combs."
They are all compatible with a well-defined causal order!

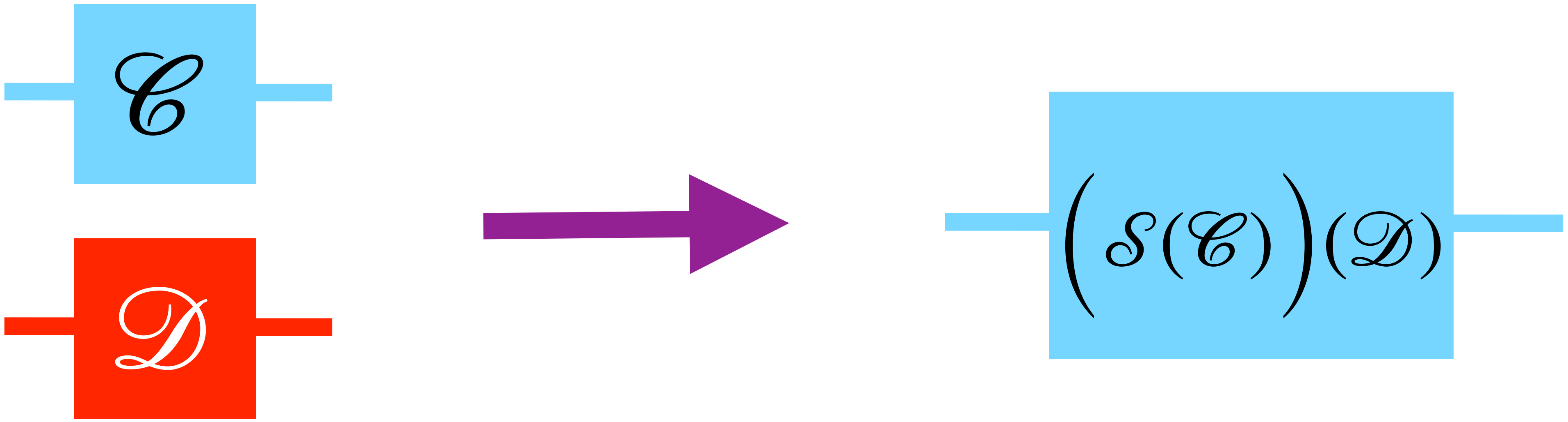
QUANTUM SUPERMAPS WITH INDEFINITE CAUSAL ORDER

FROM DEFINITE TO INDEFINITE CAUSAL ORDER

Question: what is the most general way to transform a quantum channel into a supermap?

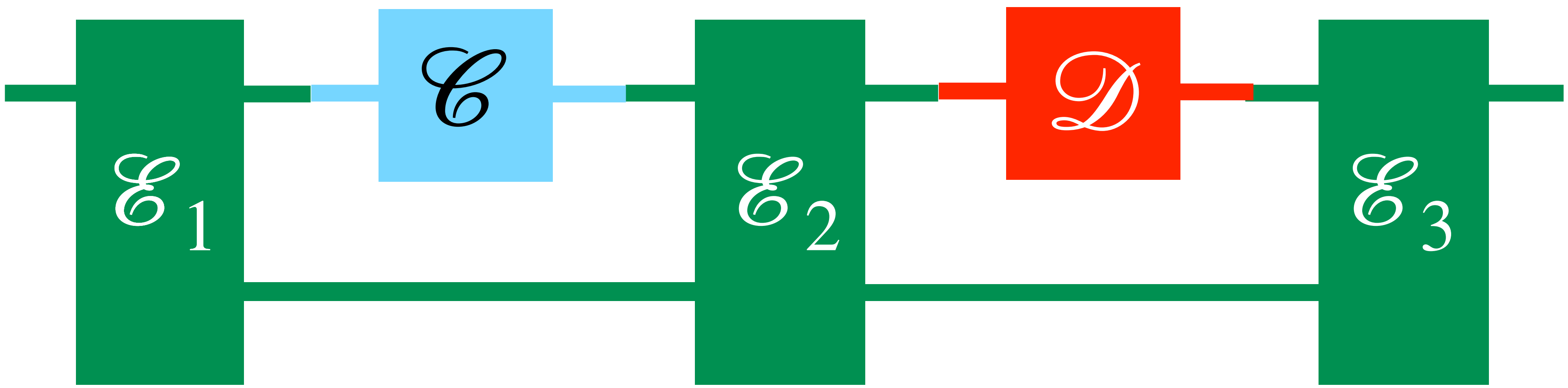


Equivalently: what is the most general way to transform a *pair* of channels into a channel?

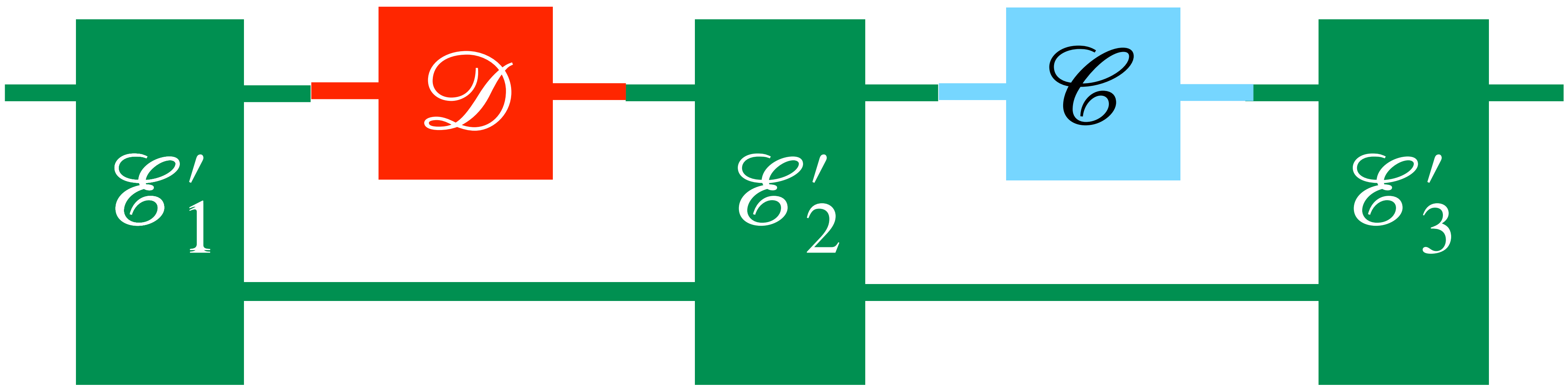


CLASSICALLY: TWO COMPLEMENTARY ORDERS

- Option 1: place $\mathscr{C}$ before $\mathscr{D}$



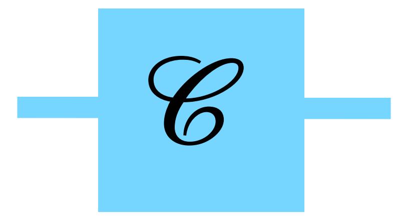
- Option 2: place $\mathscr{D}$ before $\mathscr{C}$

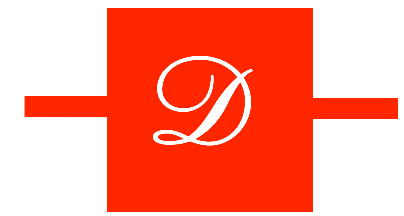


In quantum theory, however, more options are in principle possible.

THE (SIMPLIFIED) QUANTUM SWITCH

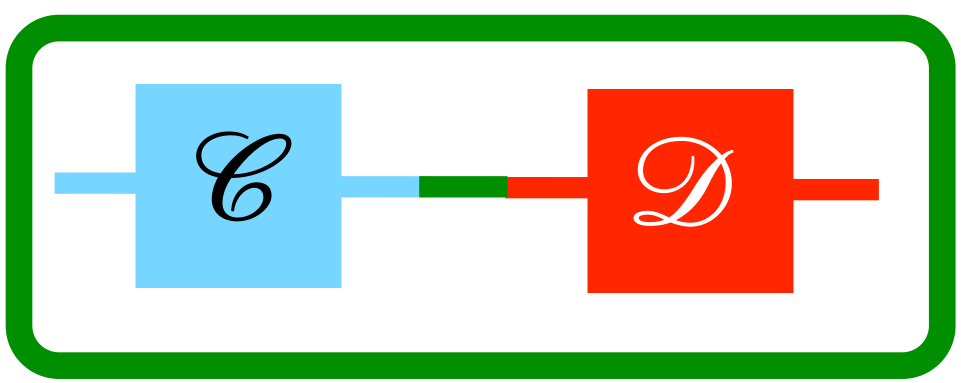
The *(simplified)* quantum SWITCH is the supermap that

takes as input the two processes 

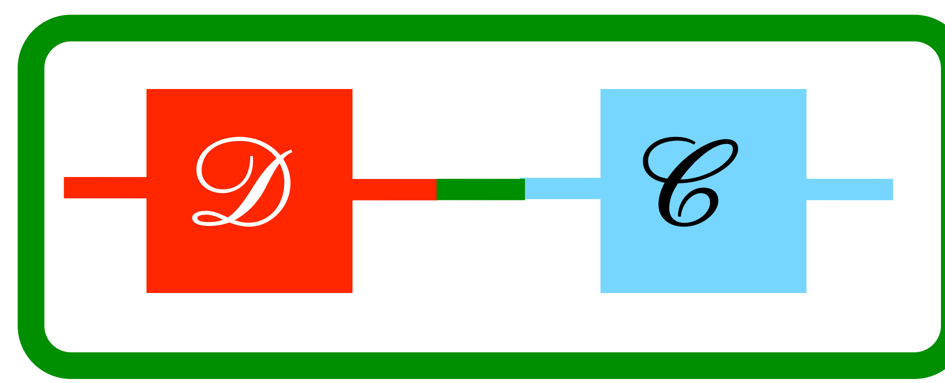
and 

with equal inputs/outputs

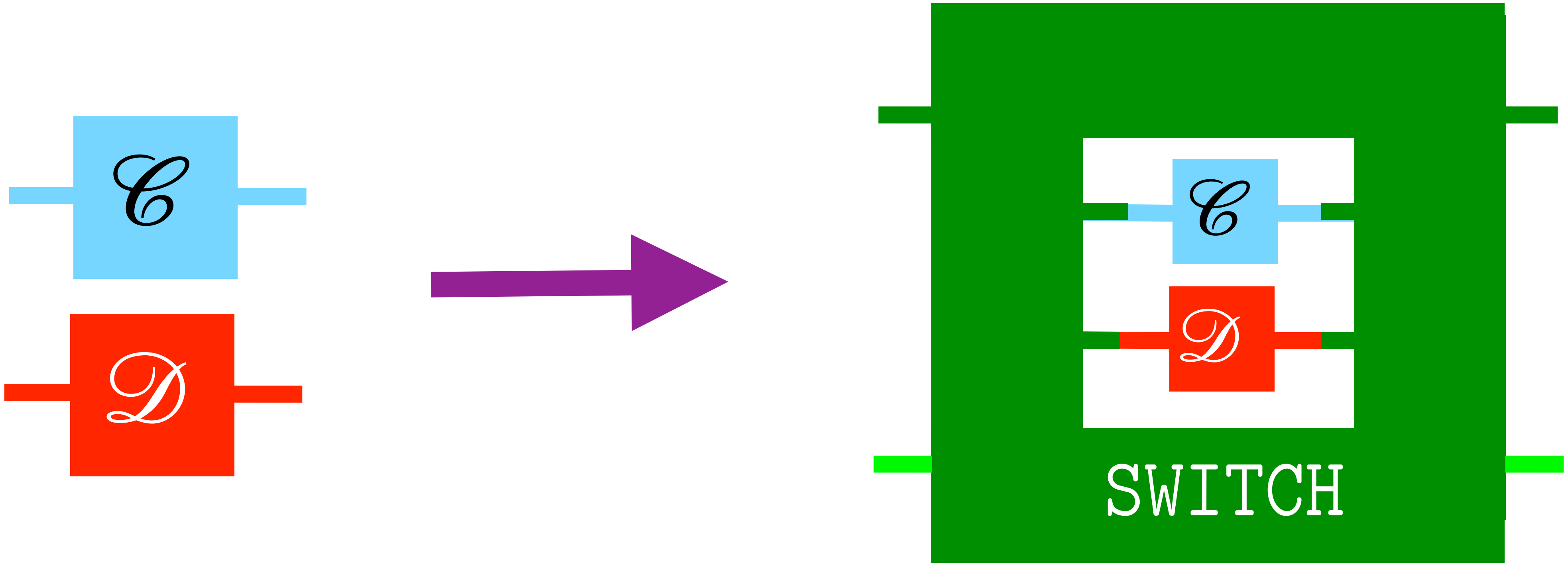
and *connects* them in a coherent superposition of the two configurations



and



Chiribella, D'Ariano, Perinotti, Valiron, arXiv:0912.0195
Phys. Rev. A 88, 022318 (2013)



$$\left[\mathtt{SWITCH}(\mathscr{C},\mathscr{D})\right](\rho) = \sum_{i,j} S_{ij}\rho S_{ij}^{\dagger}$$

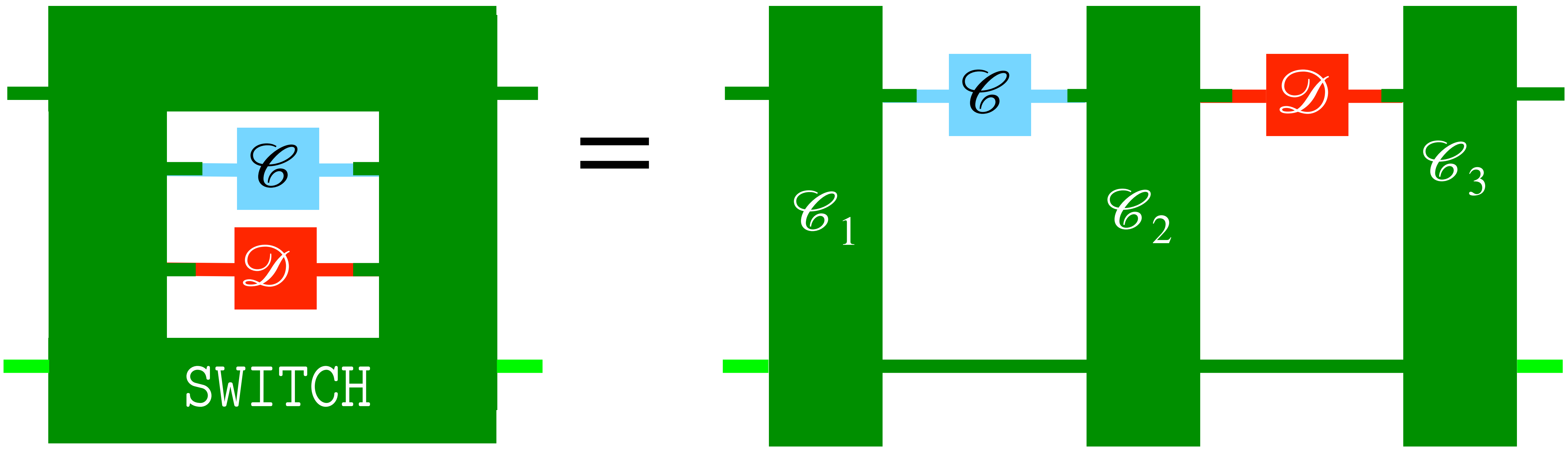
$$S_{ij} := C_i D_j \otimes |0\rangle\langle 0| + D_j C_i \otimes |1\rangle\langle 1|$$

Remark: the quantum channel $\mathtt{SWITCH}(\mathscr{C},\mathscr{D})$ is independent of the choice of Kraus operators for $\mathscr{C}$ and $\mathscr{D}$ in the above equation.

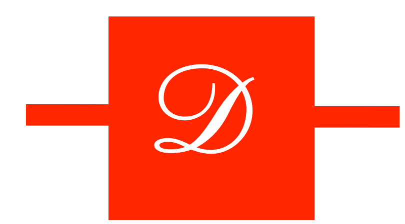
INCOMPATIBILITY WITH FIXED CAUSAL ORDER

Theorem (CDPV 2009/2013)

It is impossible to find quantum channels $\mathscr{C}_1$, $\mathscr{C}_2$, and $\mathscr{C}_3$ such that



for all unitary 

and 

The impossibility of realizing a supermap as a (random mixture of) circuits with definite order is called *causal non-separability.*

Oreshkov, Costa, Brukner, Nature Communications 3, 1092 (2012)

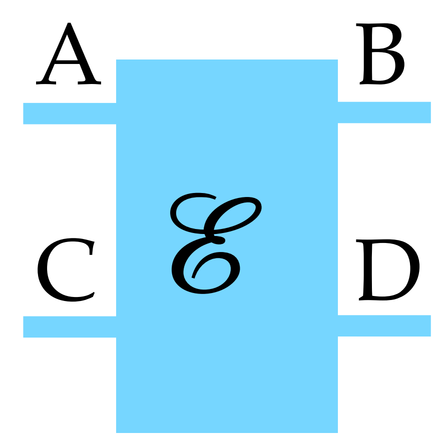
GENERAL SUPERMAPS
ON
SUBSETS OF
QUANTUM CHANNELS

GENERAL DEFINITION OF SUPERMAP

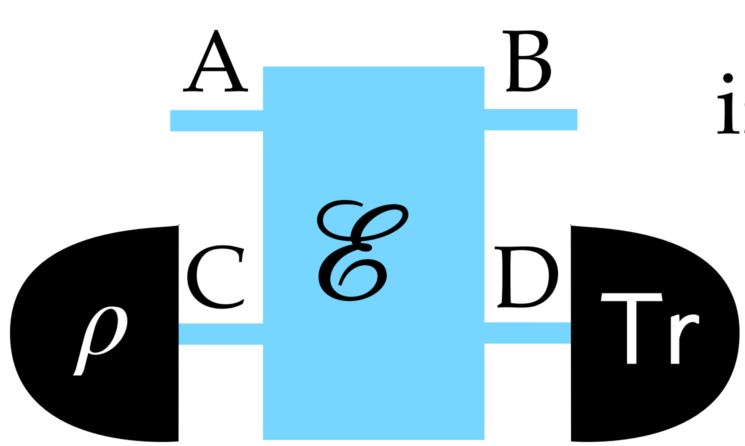
Chiribella, D'Ariano, Perinotti, Valiron, Phys. Rev. A 88, 022318 (2013)

Let S_{AB} and $\mathsf{S}'_{A'B'}$ be two subsets of (possibly multipartite) quantum channels.
A deterministic supermap from S_{AB} to $\mathsf{S}'_{A'B'}$ is a linear map that transforms channels in the *extensions** of S_{AB} into channels in the *extensions** of $\mathsf{S}'_{A'B'}$.

*an *extension* of S_{AB} is a set of channels



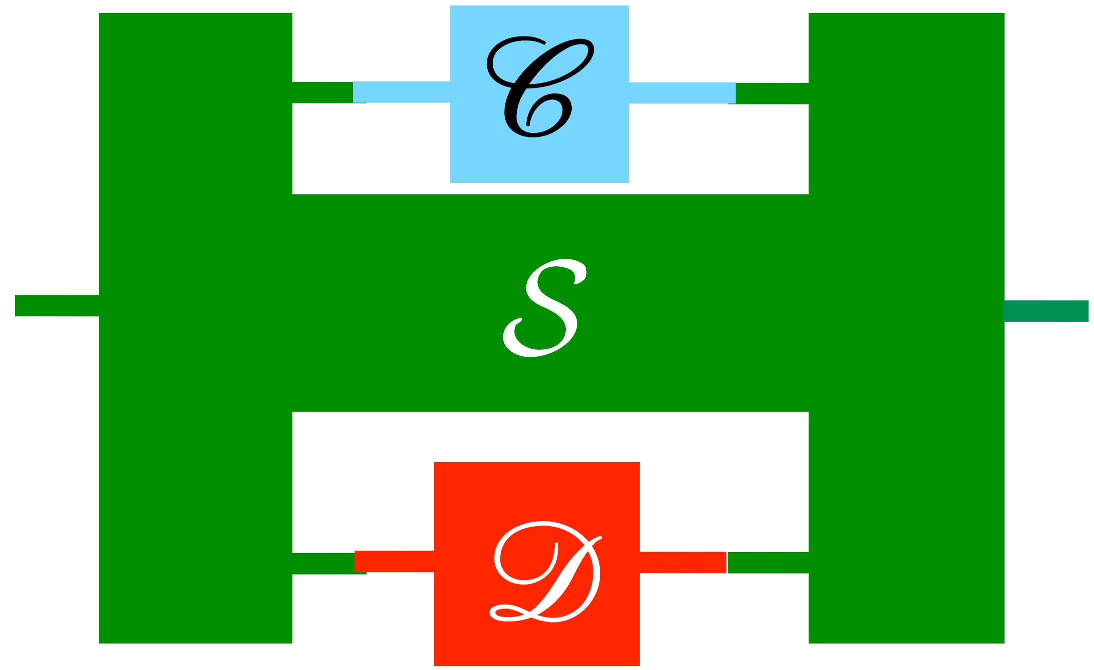
such that



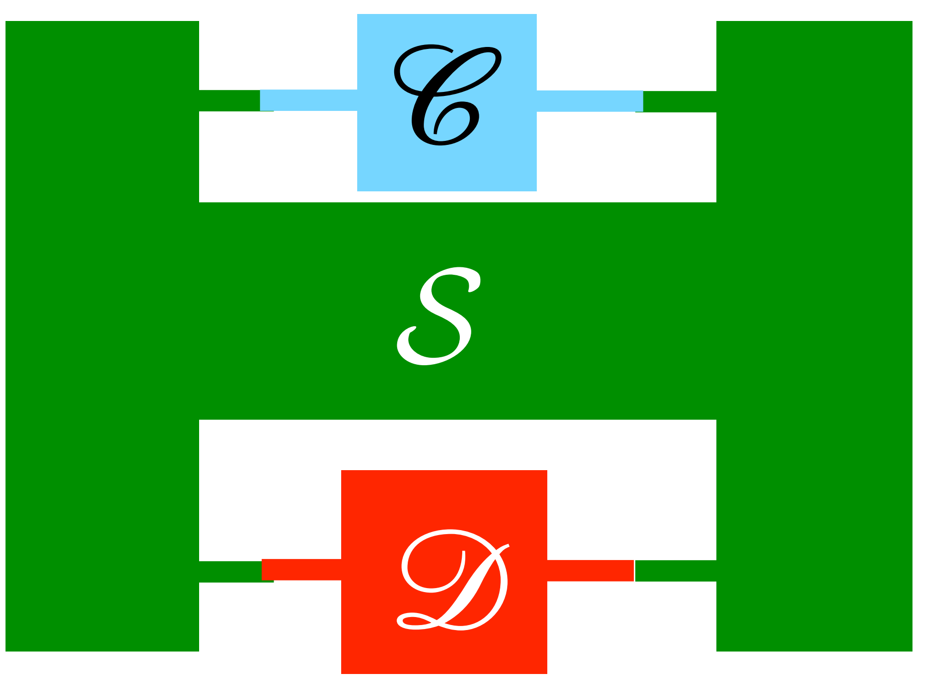
is in S_{AB} for every state ρ

SUPERMAPS ON PRODUCT CHANNELS

Supermaps from product channels to channels
e.g. the quantum switch



Special case:
supermaps from product channels to numbers.
The Choi operators of these maps are known as *process matrices.*



Oreshkov, Costa, Brukner, Nature Communications 3, 1092 (2012)

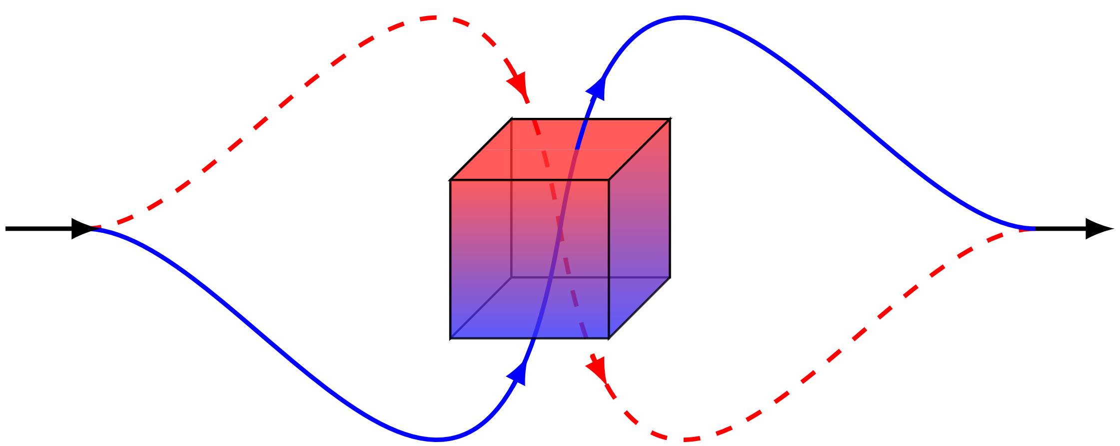
SUPERMAPS ON BISTOCHASTIC CHANNELS

Bistochastic channels = unital trace-preserving CP maps.
They constitute a time-symmetric fragment of quantum theory.

Example of supermap on bistochastic channels:

$$\left[\mathtt{FLIP}(\mathscr{C})\right](\rho) = \sum_{i,j} F_i \rho F_i^\dagger$$

$$F_i := C_i \otimes |0\rangle\langle 0| + C_i^T \otimes |1\rangle\langle 1|$$



Called the "quantum time flip,"
generates a superposition of a process and its time reversal.
The input-output direction becomes indefinite.

Chiribella and Liu, Communication Physics 5, 190 (2022)

OUTLOOK

TAKE HOME MESSAGES

- Quantum supermaps define a broad class of processes in principle compatible with quantum theory.

- Tight relation between quantum supermaps and quantum casual structures.

- New directions: supermaps in bistochastic quantum theory, indefinite input-output direction.