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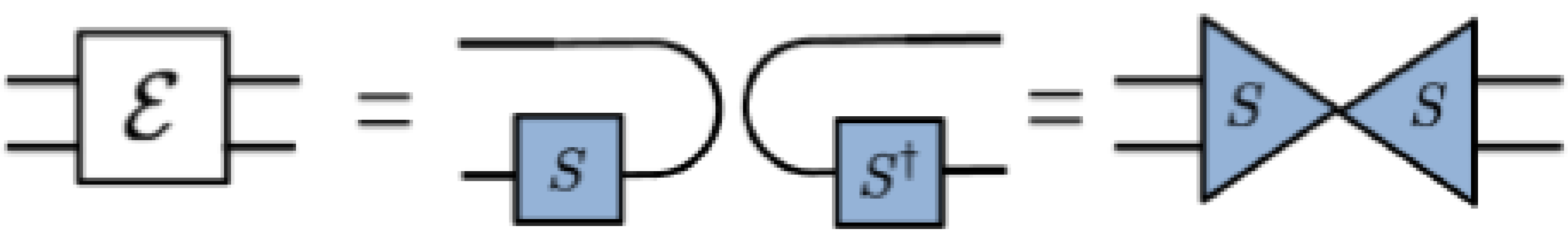
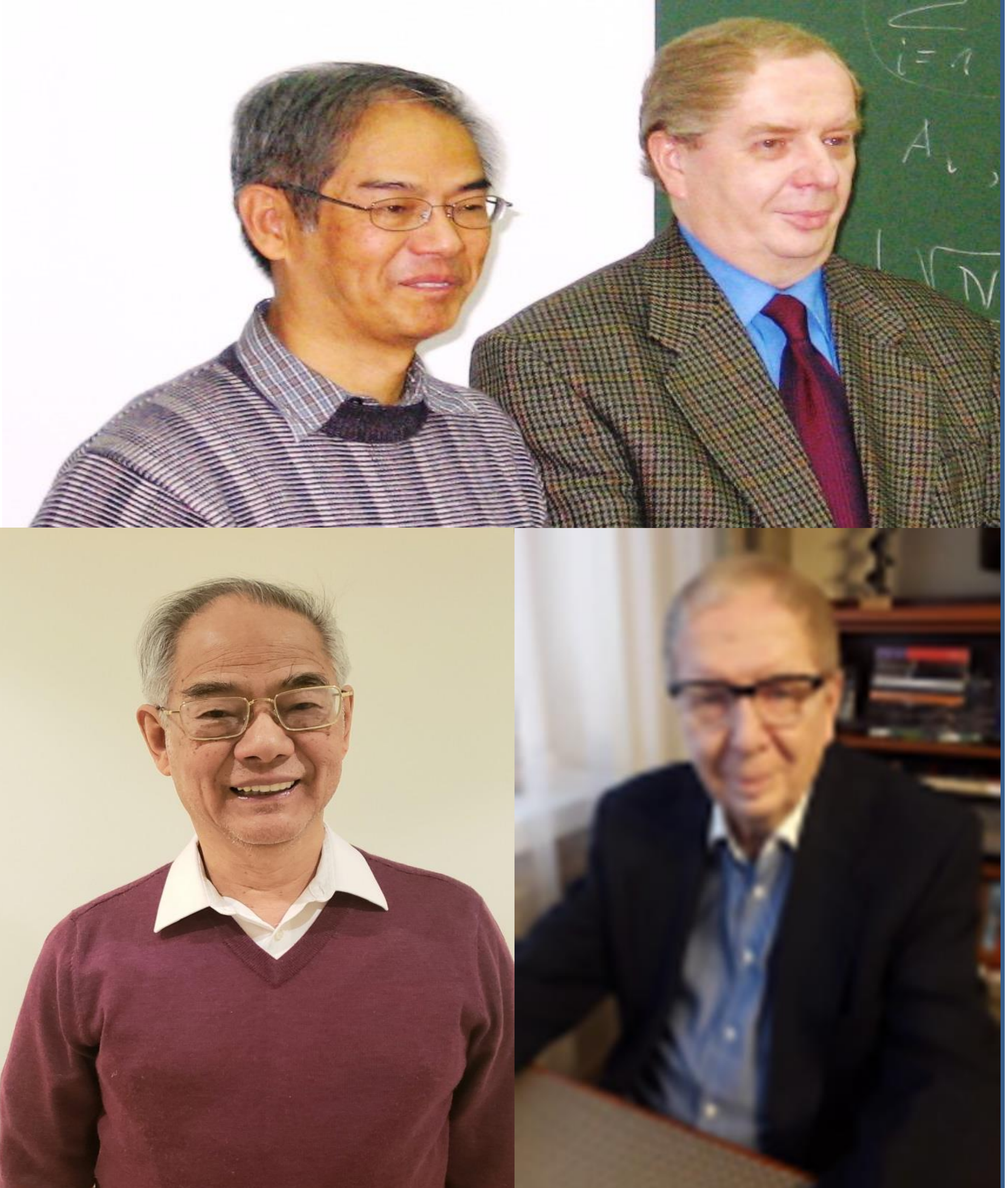
Celebrating the Choi - Jamiolkowski Isomorphism

⟨KC|K⟩

Online Event
March 1-2, 2023





From

Real Meeting

at Torun in 2008

To

Quantum Online

in 2023

Formal Title

Celebrating the

Choi - Jamiolkowski Isomorphism

Additional Sub-title

The Taming of the Shrew

Shrew = Quantum Entanglements

of Positive Semi-Definite Matrices

Who's Afraid of

Quantum Entanglements?

Tensor- product setup for the Taming of the Shrew

- Consider a Hilbert space

  $$H = H_1 \otimes H_2 .$$

- Some natural / simple / easy phenomena on H could be entangled in H_1 and H_2 separately.

- We wish to control the whole situation, bypassing / conquering /ignoring the entanglements.

Math Settings

- $L^2(X\mathbf{x}Y) = L^2(X) \otimes L^2(Y).$
- Often consider of finite-dimensional Hilbert spaces as $\boldsymbol{C}^n$ with a positive integer n.

➢ Thus $\boldsymbol{C}^n \otimes \boldsymbol{C}^k = \boldsymbol{C}^{nk}$.

$M_n =$ linear maps from $\boldsymbol{C}^n$ to $\boldsymbol{C}^n$

$$M_n \otimes M_k = M_{nk} = M_n(M_k) = M_k(M_n) .$$

--- no need to mention of anything as the universal property.

➢ In such an easy mathematical setting, who is afraid of quantum entanglements and local-global effects with respect to $\bigotimes$

Math Settings $M_n \otimes M_k = M_{nk}$ (with n>1, k>1)

➢ *{ the sums of $A_j \otimes B_j$ with A_j in M_n^+, B_j in M_k^+}* *is only a proper subset of* $(M_n \otimes M_k)^+ = M_{nk}^+$.

Reason: M_n^+ = { positive linear combinations of rank-1 projections }

•There are many rank-1 projections in M_{nk} which are not tensor product of rank-1 projections.

➢ Along this line, completely positive linear maps can go through the quantum entanglements, while positive linear maps cannot.

Quantum Entanglements provide exciting features for positive linear maps

Structure Theory

Notation: Each linear map $\varphi : M_n \to M_k$ can be extended to a linear map

$$\varphi \otimes id_p: M_n \otimes M_p \longrightarrow M_k \otimes M_p \ .$$

Def: φ is said to be ***p-positive*** when $\Phi \otimes \mathrm{id}_p$ is a positive linear map.

Def: φ is said to be ***completely positive*** when φ is a p-positive linear map for each positive integer p.

Structure Theory

Thm (Choi) : All p-positive linear maps from M_n to M_k are completely positive when $n \leq p$ or $k \leq p$.

- Nevertheless, various p provide distinct classes of p-positive linear maps as elaborated in the following:

Example (Choi): The linear map $\varphi : M_n \to M_n$ defined as $\varphi(A) = (n\text{-}1)(trace\ A)I_n - A$ is (n-1)-positve but not n-positive.

Main Thm: (Choi, 1975) A linear map

$\varphi : M_n \to M_k$ is ***completely positive***

iff $[\varphi(E_{ij})]_{i,j}$ is positive

where $\{ E_{ij} \}$ are the matrix units

iff $\varphi(A) = \Sigma V_j^* A V_j$ for all $A \in M_n$

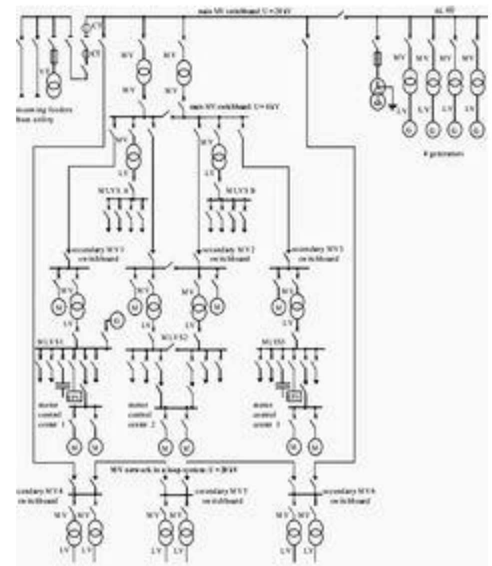
with n x k matrices V_j

- This 1975 paper (6 pages) has been cited in more than 2700 research papers, as of 2023 March

Google Scholar

- More than 1500 citations in recent publications of Quantum Information.

CIRCUIT THEORY



- Each transformer defines a

  positive linear map $A \rightarrow V^*AV$.

  Thus several transformers in series

define a completely positive linear map.

- Main concern in circuit theory: General linear maps of mathematical expressions in terms of $[\varphi(E_{ij})]_{i,j}$ are not implementable.

➢ **Classical computer vs Quantum computer**

❖ A classical computers produces 0-1 sequences

while a quantum computers produces psd matrices.

Thus only completely positive maps are usable to connect *Quantum computers*

The Main Thm (Choi 1975) revisited

Let $\varphi : M_n \to M_k$ be a linear map. TFAE:

(1) φ is p-positive for all positive integer p.

(2) $[\varphi(E_{ij})]_{i,j}$ is positive

(3) $\varphi(A) = \Sigma V_j{}^* A V_j$ for all $A \in M_n$ with n x k matrices V_j

- (1) means to be the hardest nature to conquer all incredible quantum entanglements in $(M_n \otimes M_p)^+$ of various *p*.
  (2) is intended for the simplest mathematical expression of a general linear map.
  (3) turns to be the only possible connection in circuit theory.

- Stinespring Theorem (1955) covers the case (1) ⇔ (3).

- Theorem1975 says much about (2) ⇔ (3) and (2) ⇔ (1), which is most needed in theory of quantum information.

Taming of Shrews

- NO way to describe so many incredible *entanglements* in $(M_n \otimes M_p)^+$ of various p.

- The most outstanding $T = \Sigma E_{ij} \otimes E_{ij} \in (M_n \otimes M_n)^+$, is *a* well behaved *entanglement* which serves as the representative for ALLwild entanglements.

- THEOREM says that to tame ALL shrews (= entanglements) is equivalent to tame a single LOVELY shrew (without worrying how nasty/dirty/undisciplined of other shrews).

CLOSE-UP of the LOVELY *Shrew*

Example n =3, $T = \sum E_{ij} \otimes E_{ij} \in (M_3 \otimes M_3)^+ = M_9^+$

❖ T is the NATURAL assemblage of matrix units

$$\left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]$$

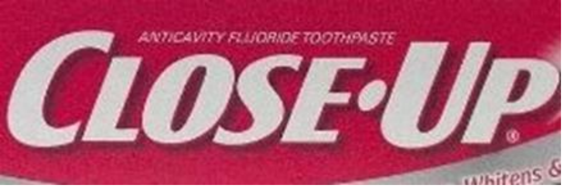
➢ Indeed, $T^2 = nT$, so $\frac{1}{n}T$ is a rank-1 projection, but T serves as the best witness to test all completely positive linear maps $M_3 \rightarrow M_3$.

Why Not Down to *n=2*?

- The simplest example of **quantum entanglement** is

$$\begin{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix}$$

as a positive 4 x 4 matrix, but not of the form as the sum of $A_j \otimes B_j$ with A_j in $M_2{}^+$ and B_j in $M_2{}^+$.



Purpose: Wish to classify all linear maps

$\varphi : M_2 \to M_2$ by means of the 4 x 4 Choi Matrix $C\varphi$

$$\begin{bmatrix} \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) & \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \\ \varphi\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) & \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) \end{bmatrix},$$

Challenge: *What sort of non-commutative geometry could be hidden/shown in the 4 x4 matrix* $C\varphi$ *?*

Newest Classification Theorem

(Joint work with C.K. Li, 2023)

Consider all $\varphi : M_2 \rightarrow M_2$ *as* unital trace-preserving and hermitian- preserving linear maps.

Then the 4 real eigenvalues of the Choi Matrix C_φ determine *the linear map* φ up to unitary equivalence.

*I.e., **iff** C_φ and C_ψ have the same eigenvalues, then there exist unitaries U and W such that*

$\varphi(A) = U^* \Psi (W^* A W) U$ for all A in M_2.

The Most Important Example:

By means of Pauli Matrices

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$,

and 4 real numbers λ_j *with* $\sum \lambda_j = 1$.

Define $\varphi : M_2 \rightarrow M_2$

as $\varphi(A) = \lambda_1 A + \lambda_2 ZAZ + \lambda_3 XAX + \lambda_4 YAY$

- Then φ is a unital linear map preserving traces and hermitian matrices.
- *The Choi Matrix* $C\varphi$ has $\{2\lambda_j\}$ as four eigenvalues.

Newest Classification Theorem Restated

Each unital qubit channel φ

(unital trace preserving completely positive linear map $M_2 \to M_2$)

is unitarily equivalent to a **concrete** map of the form $A \to \lambda_1 A + \lambda_2 ZAZ + \lambda_3 XAX + \lambda_4 YAY,$

where X, Y and Z are Pauli Matrices;

$\{2\lambda_j\}$ *are eigenvalues of the Choi Matrix* $C\varphi$.

- *This provides the WHOLE picture of unital qubit channels.*

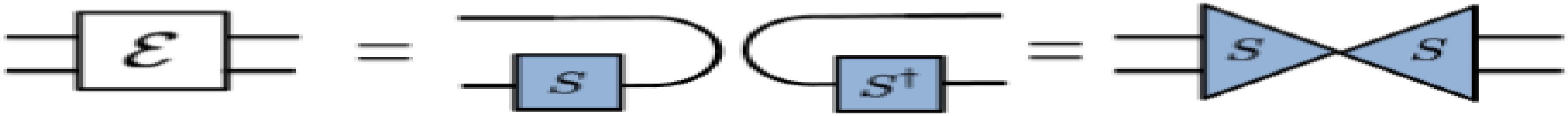
OPEN QUESTION

What would be next Classification Theorems ?

Want to study the
case n=3.

- Need to understand the quantum entanglement of

$$\left[\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1
\end{array}\right]$$



This completes my Celebration of

The C-J Isomorphism.