SOME REMARKS ON THE BLOCK POSITIVE AND POSITIVE MAPS IN PHYSICS

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My first publications

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A SPINORIAL APPROACH TO PALATINI VARIATIONAL PRINCIPLES

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The condition $\nabla_\mu g^{\alpha\dot{A}B}=0$ postulated in spinor analysis is shown to be deducible from a variational principle of the Palatini type. A generalized action principle which yields the Einstein equations, the Maxwell equations and the algebraic relations among derivatives of the metric quantities and connection quantities is studied. An action which leads to the Einstein equations with the energy-momentum tensor of a scalar massless field is investigated.

1. Introduction

The relativity theory identifies space-time with a four-dimensional differentiable manifold M_4; the set of the local maps on M_4 etablishes a correspondence between points $p\in M_4$ and four real coordinates of the points p, x^α ($\alpha=0, 1, 2, 3$). The group of coordinate transformations will be denoted by C_∞.

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LINEAR TRANSFORMATIONS WHICH PRESERVE TRACE AND POSITIVE SEMIDEFINITENESS OF OPERATORS

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This work may be considered a completion of the paper by J. de Pillis: *Linear transformations which preserve Hermitian and positive semidefinite operators*, published in 1967 [2]: necessary conditions have been formulated.

Let $\mathscr{A}_1$ be the full algebra of linear operators on the n-dimensional Hilbert space $\mathscr{H}_1$, and let $\mathscr{A}_2$ be the full algebra of linear operators on the m-dimensional Hilbert space $\mathscr{H}_2$. Let $\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ denote the complex space of linear maps from $\mathscr{A}_1$ to $\mathscr{A}_2$ and S denotes the cone of all $T\in\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ which send positive semidefinite operators from $\mathscr{A}_1$ to positive semidefinite operators from $\mathscr{A}_2$. The aim of this paper is to present a necessary and sufficient condition for a transformation in $\mathscr{L}(\mathscr{A}_1, \mathscr{A}_2)$ to be in the cone S, and to preserve trace of the operators.

1. Preliminaries and notation

Let $\mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be the vector space of linear transformations from the Hilbert space $\mathscr{H}_1$ with the inner product $(\cdot,\cdot)_1$, to the Hilbert space $\mathscr{H}_2$ with the inner product $(\cdot,\cdot)_2(\cdot,\cdot)_1$.

Preliminaries

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QUANTUM INFORMATION THEORY

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A conceptual analysis of the classical information theory of Shannon (1948) shows that this theory cannot be directly generalized to the usual quantum case. The reason is that in the usual quantum mechanics of closed systems there is no general concept of joint and conditional probability. Using, however, the generalized quantum mechanics of open systems (A. Kossakowski 1972) and the generalized concept of observable ("semiobservable", E. B. Davies and J. T. Lewis 1970) it is possible to construct a quantum information theory being then a straightforward generalization of Shannon's theory.

1. Introduction

Information theory, as it is understood in this paper and as it is usually understood by mathematicians and engineers following the pioneer paper of Shannon [57], is not only a theory of the entropy concept itself (in this aspect information theory is most interesting for physicists), but also a theory of transmission and coding of information, i.e., a theory of information sources and channels. In the case of classical (i.e., non-quantum) systems both parts of the theory are closely connected, this connection being actually accomplished in probability theory forming a theoretical background of information theory. The clue concepts are those of joint and conditional probability which enable to formulate the definition of information sources and channels and then of the concept of channel capacity which is the most important for Shannon's coding theorems. In the

Positive Maps

During the past five decades studies of positive maps represented one of the most active and fertile research topics in matrix algebra and, more generally, in C^*-algebra theory. In recent years, there have been many papers which studied duality between linear operators on the tensor product of two Hilbert spaces $\mathcal{H}_1 \otimes \mathcal{H}_2$ and linear maps that send elements of the algebra $\mathcal{B}(\mathcal{H}_1)$ of linear operators on $\mathcal{H}_1$ to elements of $\mathcal{B}(\mathcal{H}_2)$,

$$\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2) \tag{1}$$

Very often the motivation for such investigations comes from physics, in particular from the theory of open quantum systems and from quantum information theory.

Positive Maps

The main objective of my presentation is to discuss some properties of linear maps which are positive but not completely positive (CP) and the structure of such maps.

These maps correspond to the so-called entanglement witnesses — special observables defined on Hilbert spaces of composed quantum systems. For simplicity we will restrict ourselves to the simplest case of composite systems: bipartite systems of finite, but otherwise arbitrary dimensions. States of such systems are, in general, mixed and are described by density matrices.

Positive Maps

Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ be the full algebra of linear operators on N-dimensional Hilbert space $\mathcal{H}$. If $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, then the algebras $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_2)$ are isomorphic. The inner product in the vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is defined by

$$((x_1 \otimes y_1, x_2 \otimes y_2)) := (x_1, x_2)_1 \cdot (y_1, y_2)_2 \tag{2}$$

for all $x_1, x_2 \in \mathcal{H}_1$ and $y_1, y_2 \in \mathcal{H}_2$, and extended by linearity for general expressions in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Symbols $(\cdot\,,\,\cdot)_1$ and $(\cdot\,,\,\cdot)_2$ denote inner products in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Positive Maps

One defines the inner product in the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ which gives analogously the Hilbert space structure

$$[[A_1 \otimes B_1,\ A_2 \otimes B_2]] \ := \ [A_1,\ A_2]_1 \cdot [B_1,\ B_2]_2 \tag{3}$$

for all $A_1, A_2 \in \mathcal{A}_1 = \mathcal{B}(\mathcal{H}_1)$ and all $B_1, B_2 \in \mathcal{A}_2 = \mathcal{B}(\mathcal{H}_2)$. Symbols $[\,\cdot\,,\ \cdot\,]_1$ and $[\,\cdot\,,\ \cdot\,]_2$ stand for inner products in $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively, and $[X,\,Y]_i := \mathrm{tr}_i(XY^*)$.

It is well known that if a linear map $\Phi : \mathcal{A} \to \mathcal{A}$ sends the set $\mathcal{A}_* = \{X \in \mathcal{A} : \; X = X^*\}$ of all hermitian elements of $\mathcal{A}$ into itself, then Φ can be represented in the form

$$\Phi(X) \;=\; \sum_{i=1}^{\kappa} a_i K_i X K_i^* \,, \tag{4}$$

where $K_i \in \mathcal{A}$, and a_i, $i = 1, \ldots, \kappa$, are real numbers.

In general, all maps of the above form are hermiticity preserving, yet this representation is not unique: typically for a given Φ there exist many possible representations of the form (4). The smallest κ in (4) is called the minimal length of Φ. If we assume that the operators K_i, for $i = 1, \ldots, \kappa$, are linearly independent, then κ in (4) must be minimal.

k-Positive Maps

Recall that a map $\Phi : \mathcal{A} \to \mathcal{A}$ which preserves the set $\mathcal{A}_*$ of hermitian elements is called positive if $\Phi(X) \geqslant 0$ whenever $X \in \mathcal{A}$ is positive, i.e. $(X\eta, \eta) \geqslant 0$ for all $\eta \in \mathcal{H}$.

k-Positive maps

A map Φ is called *k-positive* if its k-amplification $\Phi(k) := \mathbb{1}_k \otimes \Phi$, that is the map

$$\mathbb{1}_k \otimes \Phi : M_k(\mathbb{C}) \otimes \mathcal{A} \to M_k(\mathbb{C}) \otimes \mathcal{A} \tag{5}$$

is positive.

$M_k(\mathbb{C})$ denotes here the set of all $k \times k$ complex matrices. We can identify $M_k(\mathbb{C}) \otimes \mathcal{A}$ with the set of all $k \times k$ matrices $M_k(\mathcal{A})$ with entries from $\mathcal{A}$.

Definition

Φ is called completely positive if it is k-positive for all $k = 1, 2, \ldots$.

This terminology goes back to Stinespring. It is well known that for $\mathcal{A} = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ denotes N-dimensional Hilbert space, N-positive maps on $\mathcal{A}$ are already completely positive, (M. D. Choi).

Let us observe that all hermitian-preserving maps which are not only positive but completely positive can be written in the form (4) with positive a_i, $i = 1, \ldots \kappa$, i.e. by

$$\Phi(X) \ := \sum_{i=1}^{\kappa} \tilde{K}_i X \tilde{K}_i^* \,, \tag{6}$$

where $\tilde{K}_i = \sqrt{a_i} K_i$ and $\kappa \leqslant N^2$. Relation (6) is the so-called Kraus representation of a completely positive map Φ.

Completely Positive Maps

Kraus representation is very useful in quantum information theory. In particular, CP maps are used to describe the so-called quantum operations and quantum channels. In general, any map which is positive but not completely positive can be represented as a difference of two CP maps

$$\Phi(X) = \sum_{i=1}^{\kappa_1} K_i X K_i^* - \sum_{j=1}^{\kappa_2} M_j X M_j^* , \tag{7}$$

where operators $K_1, \ldots, K_{\kappa_1}$, $M_1, \ldots, M_{\kappa_2}$ are linearly independent and

$$\kappa = \kappa_1 + \kappa_2 \tag{8}$$

denotes the minimal length of Φ.

It was shown by R. Timoney that any map Φ of the form (7) which is p-positive, where $p = [\sqrt{\kappa}\,]$, must be completely positive.

In fact, there exists a close relationship between positive maps which are not completely positive and the concept of entanglement witnesses, i.e. observables (self-adjoint operators) on Hilbert spaces of composite systems that permit to detect the presence of entangled states.

Entanglement witness

An operator $W \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is an entanglement witness if it fulfils the following conditions

(i) $((x \otimes y,\ Wx \otimes y)) \geqslant 0$ for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$,

(ii) there exists $\eta \in \mathcal{H}_1 \otimes \mathcal{H}_2$ such that $((\eta,\ W\eta)) < 0$.

Entanglement Witnesses

In simple words, an entanglement witness, not being a positive operator itself, is positive on product states (in quantum-information terminology: on separable pure states).

Recall that there exists a one-to-one correspondence between positive maps $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ and hermitian operators W on $\mathcal{H}_1 \otimes \mathcal{H}_2$ that fulfil condition (i), $((x \otimes y,\, Wx \otimes y)) \geqslant 0$. In other words, there exists an isomorphism

$$\Omega : \mathcal{B}(\mathcal{A}_1, \mathcal{A}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \tag{9}$$

defined by

$$\Omega(\Phi) \;:=\; \sum_i E_i \otimes \Phi(E_i)\,, \tag{10}$$

where $\Phi : \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ and $\{E_i\}_{i=1}^{N^2}$ is any orthonormal basis in $\mathcal{A}_1 = \mathcal{B}(\mathcal{H}_1)$.

Denoting $W_\Phi = \Omega(\Phi)$ we can say that a linear map $\Phi \in \mathcal{B}(\mathcal{A}_1, \mathcal{A}_2)$ transforms hermitian operators in $\mathcal{A}_1$ to hermitian operators in $\mathcal{A}_2$ if and only if the operator W_Φ is hermitian on $\mathcal{H}_1 \otimes \mathcal{H}_2$.

Analogously, it was proved more than five decades ago that a linear hermitian map $\Phi(X) = \sum_{i=1}^{\kappa} a_i K_i X K_i^*$ sends positive operators in $\mathcal{A}_1$ to positive operators in $\mathcal{A}_2$ if and only if the operator W_Φ fulfils

$$((x \otimes y\,, W_\Phi x \otimes y)) \geqslant 0 \tag{11}$$

for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$.

Entanglement Witnesses

Of course, the condition $((x \otimes y\,,\,W_\Phi x \otimes y)) \geqslant 0$ is weaker than the condition for positive semi-definiteness on $\mathcal{H}_1 \otimes \mathcal{H}_2$, since vectors of the form $x \otimes y$ do not constitute the whole vector space $\mathcal{H}_1 \otimes \mathcal{H}_2$. If we have $((\eta,\ W_\Phi \eta)) \geqslant 0$ for all $\eta \in \mathcal{H}_1 \otimes \mathcal{H}_2$, then Φ is completely positive.

This means that observables that are entanglement witnesses correspond to positive maps which are not completely positive or, on the base of the previous discussion, have the form

$$\Phi(X) = \sum_{i=1}^{\kappa_1} K_i X K_i^* \; - \; \sum_{j=1}^{\kappa_2} M_j X M_j^* \,,$$

with minimal length $\kappa_1 + \kappa_2$.

The Structure of CP Maps

Let $M_n(\mathbb{C})$ be the space of $n \times n$ complex matrices. Examining the expression

$$\Phi(X) = \sum_{i=1}^{\kappa} A_i X A_i^* \tag{12}$$

we see that every CP map can be associated with the subalgebra of $M_n(\mathbb{C})$, namely, the algebra $\mathcal{A}(A_1, \ldots, A_\kappa)$ generated by $A_1, \ldots, A_\kappa$. Intuitively, this algebra contains all expressions in the form $A_{i_2}^{n_1} A_{i_2}^{n_1} \cdots A_{i_\kappa}^{n_\kappa}$ and their linear combinations. This algebra is independent of the particular representation (12), and thus we can also use the notation $\mathcal{A}(\Phi)$. Every quantum map has to be positive, i.e. it has to preserve the positive cone $M_n^+(\mathbb{C})$ in $M_n(\mathbb{C})$.

Irreducible maps

A CP map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called irreducible if there exists no nontrivial face of the cone $M_n^+(\mathbb{C})$ invariant under Φ.

If the map Φ is given by its Kraus decomposition, the above definition can be expressed in an equivalent way:

Irreducible maps

A CP map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by

$$\Phi(X) = \sum_{i=1}^{\kappa} A_i X A_i^*$$

is irreducible iff the operators $A_1, \ldots, A_\kappa$ have no nontrivial common invariant subspace.

Trivial subspaces are $\{0\}$ and $\mathbb{C}^n$.

The Structure of CP Maps

Now, we will make use of the classical result stating that if a given algebra $\mathcal{A}(\Phi)$ is a *-algebra, i.e. it is closed under Hermitian conjugation, then one can choose an orthonormal basis in which $\mathcal{A}$ is block-diagonal (Barker, Eifler, Kezlan, 1978).

Corollary

Let $\mathcal{H} \simeq \mathbb{C}^n$, $\mathcal{B}(\mathcal{H}) \simeq M_n(\mathbb{C})$ and let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a trace preserving or unital CP map written in the Kraus form,

$$\Phi(X) \;=\; \sum_{i=1}^{\kappa} A_i X A_i^* \,. \tag{13}$$

If $\mathcal{A}(A_1, \ldots, A_\kappa)$ is a *-algebra, then there exists an orthonormal basis $\{e_i\}_{i=1}^n$ and natural numbers $d_1, \ldots, d_N$ for which all Kraus operators have the block diagonal form, where each block A_{im} has dimension $d_m \times d_m$, $\sum_j d_j = n$ and $\mathcal{A}(A_{1m}, \ldots, A_{\kappa m}) \simeq M_{d_m}(\mathbb{C})$.

The Structure of CP Maps

Consequently, there exists a decomposition of the Hilbert space $\mathcal{H}$ such that

$$\mathcal{H} \; = \; \bigoplus_{j=1}^{N} \mathcal{H}_j \,, \tag{14}$$

where $\dim \mathcal{H}_k = d_k$ for $k = 1, \ldots, N$, $A_{im} : \mathcal{H}_m \to \mathcal{H}_m$, and

$$A_{im} \; = \; A_i|_{\mathcal{H}_m} \,. \tag{15}$$

The most important examples of the maps for which the algebra $A(\Phi)$ is a *-algebra are unital quantum channels. In essence, the above Corollary states that if $\mathcal{A}(\Phi)$ is a *-algebra, then one can decompose ("reduce") the Kraus operators into smaller, irreducible blocks.

Now the question is, how we can investigate the structure of the algebra $\mathcal{A}(\Phi)$, in order to check the dimensionality of the blocks in the decomposition of the algebra (we are dealing with maps for which $\mathcal{A}(\Phi)$ is a *-algebra, so such decomposition exists). There are several tools for analysing the internal structure of the algebra $\mathcal{A}(\Phi)$ using only its generators, that is, its Kraus operators. The most important are the Shemesh criterion and the Amitsur-Levitzki theorem.

Theorem (Shemesh 1984)

Matrices $A, B \in M_n(\mathbb{C})$ have a common eigenvector if and only if

$$\mathcal{M} = \bigcap_{k,l=1}^{n-1} \ker[A^k, B^l] \neq \{0\}. \tag{16}$$

It is not difficult to show that $\mathcal{M}$ is the smallest subspace of $\mathcal{H} = \mathbb{C}^n$ which contains all common eigenvectors of the matrices A and B. At the same time, the subspace $\mathcal{M}$ defined in (16) is the common invariant subspace of A and B on which they commute.

Condition (16) can be represented in an equivalent form: A and B have a common eigenvector iff

$$\det\left(\sum_{k,l=1}^{n-1} [A^k, B^l]^*[A^k, B^l]\right) = 0.$$

There exists a generalisation of the Shemesh theorem for an arbitrary number of matrices.

Theorem (Pastuszak, Jamiołkowski, 2015)

Assume that $H, A_1, \dots A_s \in M_n(\mathbb{C})$ and that H has all distinct eigenvalues. Let

$$\mathcal{N}(H, A_1, \dots, A_s) = \bigcap_{k=1}^{n-1} \bigcap_{i=1}^{s} \ker[H^k, A_i]. \tag{17}$$

Then the marices $H, A_1, \dots, A_s$ have a common eigenvector iff $\mathcal{N}(H, A_1, \dots, A_s) \neq \{0\}$.

The standard polynomial for n noncommutative variables $X_1, \ldots, X_n$ is defined in the following way:

$$S_n(X_1, \ldots, X_n) \;=\; \sum_{\sigma \in S_n} \operatorname{sign} X_{\sigma(1)} \cdots X_{\sigma(n)}\,, \tag{18}$$

where the summation runs over all permutations of $\{1, \ldots, n\}$.

Theorem (Amitsur and Levitzki, 1950)

The full matrix algebra $M_n(\mathbb{C})$ satisfies the standard polynomial identity of order $2n$, that is, for all matrices $A_1 \ldots, A_{2n}$, we have

$$S_{2n}(A_1, \ldots, A_{2n}) \;=\; 0\,. \tag{19}$$

Moreover the algebra $M_n(\mathbb{C})$ satisfies no identity of order smaller than $2n$.

The Structure of CP Maps

Let us observe that according to the above theorem the algebra $M_{d+1}(\mathbb{C})$ cannot satisfy the standard identity for $n = 2d$. In other words, the algebra $M_k(\mathbb{C})$ satisfies the identity $S_{2d} = 0$ when $k \leqslant d$, but does not satisfy it for $k \geqslant d + 1$.

The way of using the Amitsur-Levitzki theorem to analyze the structure of CP maps is discussed in:

A. Jamiołkowski, *"Applications of PI algebras to the analysis of quantum channels"*, Int. J. Quant. Inf., 2012.