Non-decomposable maps from tensor products

Alexander Müller-Hermes

University of Oslo

Fact: Detecting entanglement is hard!

Difficult problem: Given $X_{AB} \in (\mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B})^+$ determine whether

$$X_{AB} \in \operatorname{conv}\{Y_A \otimes Z_B : Y_A \in \mathcal{M}_{d_A}^+, Z_B \in \mathcal{M}_{d_B}^+\}$$

or not.

Easy solution if $(d_A, d_B) \in \{(2,2), (2,3), (3,2)\}$ [Horodecki, Størmer, Woronowicz]

Here: X_{AB} is entangled if and only if

$$X_{AB}^{\Gamma} := (\underbrace{id_A \otimes \vartheta_B}_{\text{Transpose on } \mathcal{M}_{d_B}.})(X_{AB}) \not\geq 0$$

Transpose detects low-dimensional entanglement.

For $(d_A, d_B) \notin \{(2,2), (2,3), (3,2)\}$ the transpose does not detect all entanglement!

$\leadsto$ $\exists$ entangled $X_{AB} \in (\mathcal{M}_{d_A} \otimes \mathcal{M}_{d_B})^+$ with <u>P</u>ositive <u>P</u>artial <u>T</u>ranspose (PPT)

$$\boxed{\text{What positive maps } P: \mathcal{M}_{d_B} \to \mathcal{M}_{d_A} \text{ detect PPT entanglement, i.e. s.th } (\mathrm{id}_A \otimes P)(X_{AB}) \ngeq 0 \;?}$$

$\leadsto$ P cannot be <u>decomposable</u>: $P = T_1 + \vartheta \circ T_2$ (Transpose: ϑ)

for completely positive T_1, T_2

We need a <u>positive non-decomposable map</u>!

Example (Choi): $P: M_3 \to M_3$ given by

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \longmapsto \begin{pmatrix} x_{11}+x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{22}+x_{11} & -x_{23} \\ -x_{31} & -x_{32} & x_{33}+x_{22} \end{pmatrix}$$

$\rightsquigarrow$ Structure of general non-decomposable positive maps is not well understood!

$\rightsquigarrow$ No closed form, etc ...

$\rightsquigarrow$ Would be nice to construct such maps from simpler "building blocks".

Idea: Can we build non-decomposable maps of the form $P \otimes Q$ for decomposable maps P & Q?

General question: How does decomposability behave under tensor products?

In the rest of the talk:

1. Simple trick illustrating this principle
2. Some more general results

$\underline{\text{Thm:}}$ (Piani & Mora)

If $P: M_d \to M_d$ is k-positive and <u>not</u> completely positive, then $id_k \otimes P$ is positive and not decomposable.

Example: $P: M_3 \to M_3$

$$\left.\begin{array}{l} P: M_3 \to M_3 \\ P(X) = 2\,\mathrm{Tr}[X]\,\mathbb{1}_3 - X \end{array}\right\} \begin{array}{l} \text{2-positive (due to Choi)} \\ \text{\& not CP} \end{array}$$

$\Rightarrow id_2 \otimes P: M_6 \to M_6$ is positive and non-decomposable.

Let's see a proof for $k=2$.

Proof for $k=2$: Show that $id_2 \otimes P$ not decomposable if P is not CP.

We will need an interesting matrix:

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \in (\mathcal{M}_2 \otimes \mathcal{M}_2)^+$$

$rk(A) = 3$ & $|\Omega_2\rangle = \sum_{i=1}^{2} |i\rangle \otimes |i\rangle \in \ker(A)$.

$$A^\Gamma = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \in (\mathcal{M}_2 \otimes \mathcal{M}_2)^+ \rightsquigarrow A \text{ is PPT.}$$

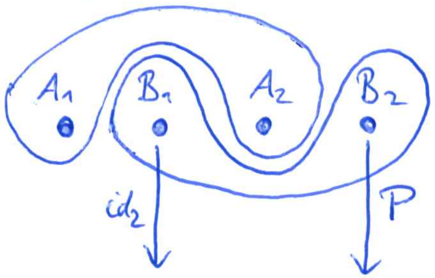
$rk(A^\Gamma) = 4 \rightsquigarrow$ full-rank.

Show that $id_2 \otimes P$ detects some PPT entanglement.

Need a PPT state:

$$\begin{cases} X_{AB} = A \otimes \mathbb{1}_d \otimes \mathbb{1}_d + \varepsilon\, |\Omega_2\rangle\langle\Omega_2| \otimes |\Omega_d\rangle\langle\Omega_d| \in \big(\underset{\substack{\uparrow \\ A_1}}{\mathcal{M}_2} \otimes \underset{\substack{\uparrow \\ B_1}}{\mathcal{M}_2} \otimes \underset{\substack{\uparrow \\ A_2}}{\mathcal{M}_d} \otimes \underset{\substack{\uparrow \\ B_2}}{\mathcal{M}_d}\big)^+ \\[2ex] X_{AB}^{\Gamma_{B_1 B_2}} = \underbrace{A^{\Gamma} \otimes \mathbb{1}_d \otimes \mathbb{1}_d}_{>0} + \varepsilon\, \mathbb{F}_2 \otimes \mathbb{F}_d \end{cases}$$

$\Longrightarrow$ For $\varepsilon > 0$ small enough X_{AB} is PPT.



$$(id_2 \otimes id_2 \otimes id_d \otimes P)(X_{AB})$$

Choi matrix.

$$= A \otimes \mathbb{1}_d \otimes P(\mathbb{1}_d) + \varepsilon\, |\Omega_2\rangle\langle\Omega_2| \otimes C_P$$

$\ngeq 0$ since $(\langle\Omega_2| \otimes \mathbb{1}_d \otimes \mathbb{1}_d) \cdots (|\Omega_2\rangle \otimes \mathbb{1}_d \otimes \mathbb{1}_d)$

$$= 2\varepsilon\, C_P \ngeq 0$$

$\Longrightarrow$ $id_2 \otimes P$ not decomposable! $\square$

Let's make a small deviation

and talk about

SCHWARZ MAPS !

Def: A unital map $P : \mathcal{M}_d \to \mathcal{M}_d$ is called <u>Schwarz</u> if

$$P(X^*X) \geq P(X)^* P(X) \qquad \forall X \in \mathcal{M}_d \qquad (*)$$

$(*)$ is equivalent to

$$(\mathrm{id}_2 \otimes P)\left(\begin{pmatrix} \mathbb{1}_d & X \\ X^* & X^*X \end{pmatrix}\right) \geq 0 \quad \forall X \in \mathcal{M}_d .$$

$\rightsquigarrow$ Consequence: All 2-positive maps are Schwarz!

Choi: $\exists$ Schwarz maps that are not 2-positive!

Some people call Schwarz maps $\frac{3}{2}$-positive as a joke.

Let's take this joke seriously!

What maps could be $\frac{3}{2}$-positive?

Maybe if $P: M_d \to M_d$ is 3-positive, then

$id_2 \otimes P$ is $\frac{3}{2}$-positive?

Amazingly, this is true!

Thm: (Carlen, AMH '22)

If $P: \mathcal{M}_d \to \mathcal{M}_d$ is unital and $(k+1)$-positive, then $id_k \otimes P$ is Schwarz

Indeed: If P is 3-positive, then $id_2 \otimes P$ is $\frac{3}{2}$-positive!

Proof ($k=2$):

For $A, B, C, D \in \mathcal{M}_d$ and $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ we have

$$\begin{pmatrix} (id_2 \otimes P)(\mathbb{1}_{2d}) & (id_2 \otimes P)(X) \\ (id_2 \otimes P)(X^*) & (id_2 \otimes P)(X^* X) \end{pmatrix} = \begin{pmatrix} P(\mathbb{1}_d) & 0 & P(A) & P(B) \\ 0 & P(\mathbb{1}_d) & P(C) & P(D) \\ P(A)^* & P(C)^* & P(A^*A + C^*C) & P(A^*B + C^*D) \\ P(B)^* & P(D)^* & P(B^*A + D^*C) & P(B^*B + D^*D) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} P(\mathbb{1}_d) & 0 & P(A) & P(B) \\ 0 & 0 & 0 & 0 \\ P(A)^* & 0 & P(A^*A) & P(A^*B) \\ P(B)^* & 0 & P(B^*A) & P(B^*B) \end{pmatrix}}_{\geq 0} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & P(\mathbb{1}_d) & P(C) & P(D) \\ 0 & P(C)^* & P(C^*C) & P(C^*D) \\ 0 & P(D)^* & P(D^*C) & P(D^*D) \end{pmatrix}}_{\geq 0} \geq 0$$

$\square$

We can use this to construct non-decomposable Schwarz maps.

Example: $P : \mathcal{M}_4 \longrightarrow \mathcal{M}_4$

$P(X) = 3\,\mathrm{Tr}[X]\,\mathbb{1}_4 - X$ $\Big\}$ 3-positive (due to Choi)

Then: $\mathrm{id}_2 \otimes P : \mathcal{M}_8 \longrightarrow \mathcal{M}_8$

is $\frac{3}{2}$-positive & not decomposable

Let's get back to the general question from before:

$$\boxed{\text{How does decomposability behave under tensor products?}}$$

Something interesting is happening already for qubit maps $P: M_2 \longrightarrow M_2$!

We can find examples $P, Q : M_2 \to M_2$ s.th.

$P \otimes Q$ is positive & not decomposable.

For example:

$$P\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \begin{pmatrix} \frac{1}{2}(x_{11}+x_{22}) & \frac{2}{3}x_{12} \\ \frac{2}{3}x_{21} & \frac{1}{2}(x_{11}+x_{22}) \end{pmatrix}$$

and

$$Q\left(\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right) = \frac{1}{4}(x_{11}+x_{22})\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} + \frac{3}{4}\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

$\rightsquigarrow$ certain depolarizing channel

But ......

.... we have the following result:

$$\boxed{\begin{array}{l}\underline{\text{Thm (AMH '21):}}\\ \text{For } P: M_2 \to M_2 \text{ the following are equivalent:}\\ \quad (1)\ P \otimes P \text{ positive}\\ \quad (2)\ P \otimes P \text{ decomposable}\end{array}}$$

Similar to well-known result (Stormer / Woronowicz)

$P: M_2 \to M_2$ positive $\Longleftrightarrow$ $P: M_2 \to M_2$ decomposable

<u>Note:</u> $\exists$ examples of decomposable maps $P: M_2 \to M_4$ s.th. $P \otimes P$ is positive and not decomposable.

$\rightsquigarrow$ Theorem seems to be specific to qubits!

How to prove this theorem?

For $P : \mathcal{M}_2 \longrightarrow \mathcal{M}_2$ we have to show that

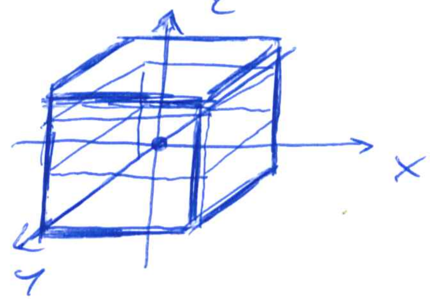
$$P \otimes P \text{ positive} \Longrightarrow P \otimes P \text{ decomposable}$$

<u>Step 1</u>: Reduce problem to Pauli-diagonal maps

$$\begin{cases} \Pi_\lambda : \mathcal{M}_2 \longrightarrow \mathcal{M}_2 \\ \Pi_\lambda(X) = \frac{1}{2}\mathrm{Tr}[X]\frac{\mathbb{1}}{2} + \frac{\lambda_1}{2}\mathrm{Tr}[\sigma_1 X]\sigma_1 + \frac{\lambda_2}{2}\mathrm{Tr}[\sigma_2 X]\sigma_2 + \frac{\lambda_3}{2}\mathrm{Tr}[\sigma_3 X]\sigma_3 \end{cases}$$

where $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

and $\lambda \in \mathbb{R}^3$ parameter vector.



$\rightsquigarrow$ Π_λ is positive iff $\lambda \in [-1, 1]^3$

Step 2: Show that

$$\Pi_\lambda \otimes \Pi_\lambda \text{ positive} \Longrightarrow \Pi_\lambda \otimes \Pi_\lambda \text{ decomposable.}$$

Note: If $\Pi_\lambda \otimes \Pi_\lambda$ is positive, then

$$(\Pi_\lambda \otimes \Pi_\lambda)(|\Omega_2\rangle\langle\Omega_2|) \geq 0 \quad \text{for} \quad |\Omega_2\rangle = |1\rangle\otimes|1\rangle + |2\rangle\otimes|2\rangle$$

$$\Updownarrow$$

$$(id_2 \otimes \Pi_\lambda^2)(|\Omega_2\rangle\langle\Omega_2|) \geq 0$$

$$\Updownarrow$$

Π_λ^2 is completely positive

$$\boxed{\begin{aligned} 1+\lambda_1^2 &\geq \lambda_2^2 + \lambda_3^2 \\ 1+\lambda_2^2 &\geq \lambda_1^2 + \lambda_3^2 \\ 1+\lambda_3^2 &\geq \lambda_1^2 + \lambda_2^2 \end{aligned}}$$

$\rightsquigarrow$ Actually those conditions also imply that $\Pi_\lambda \otimes \Pi_\lambda$ positive (Fillipov, Magadov)

Assume $\Pi_\lambda \otimes \Pi_\lambda$ positive $\rightsquigarrow \begin{cases} 1+\lambda_1^2 \geq \lambda_2^2 + \lambda_3^2 \\ 1+\lambda_2^2 \geq \lambda_1^2 + \lambda_3^2 \\ 1+\lambda_3^2 \geq \lambda_1^2 + \lambda_2^2 \end{cases}$

How to show that $\Pi_\lambda \otimes \Pi_\lambda$ is decomposable?

Observation 1: $\Pi_\lambda \otimes \Pi_\lambda = \Pi_\mu^{(2)}$ for some generalized Pauli-diagonal

$$\begin{cases} \Pi_\mu^{(2)} : \mathcal{M}_4 \longrightarrow \mathcal{M}_4 \\ \Pi_\mu^{(2)}(X) = \sum_{i_1 i_2} \frac{\mu_{i_1 i_2}}{2^2} \operatorname{Tr}\left[(\sigma_{i_1} \otimes \sigma_{i_2}) X\right] \sigma_{i_1} \otimes \sigma_{i_2} \end{cases}$$

$\rightsquigarrow \mu = \lambda \otimes \lambda$

When is a generalized Pauli-diagonal map decomposable?

Let's do this slightly more general !

When is a generalized Pauli-diagonal map

$$\begin{cases} \Pi_\mu^{(N)} : \mathcal{M}_2^{\otimes N} \to \mathcal{M}_2^{\otimes N} \\ \Pi_\mu^{(N)}(X) = \sum_{i_1 \dots i_N \in \{1,\dots,4\}} \frac{\mu_{i_1 \dots i_N}}{2^N} \operatorname{Tr}\left[(\sigma_{i_1} \otimes \dots \otimes \sigma_{i_N}) X\right] \sigma_{i_1} \otimes \dots \otimes \sigma_{i_N} \end{cases}$$

decomposable ?

$\rightsquigarrow$ In the end we will need $N=2$.

Maybe, simpler:

When is $\Pi_\mu^{(N)} : M_2^{\otimes N} \to M_2^{\otimes N}$ completely positive?

When are generalized Pauli-diagonal maps completely positive?

$$\Pi_\mu^{(N)}(X) = \sum_{i_1,\dots,i_N \in \{1,\dots,4\}^N} \frac{\mu_{i_1\dots i_N}}{2^N} \mathrm{Tr}\left[(\sigma_{i_1}\otimes\dots\otimes\sigma_{i_N})X\right]\sigma_{i_1}\otimes\dots\otimes\sigma_{i_N}$$

Choi matrix (reshuffled slightly):

$$C_{\Pi_\mu^{(N)}} = \sum_{i_1\dots i_N} \frac{\mu_{i_1\dots i_N}}{2^N} \underbrace{\sigma_{i_1}\otimes\sigma_{i_1}^T\otimes\dots\otimes\sigma_{i_N}\otimes\sigma_{i_N}^T}$$

Commuting set of operators $\rightsquigarrow$ simultaneously diagonalizable!

Thm: $\Pi_\mu^{(N)}$ is completely positive if and only if

$D^{\otimes N}\mu$ is entrywise positive

where $D = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$

$\rightsquigarrow$ generalization of Fujiwara-Algoet.

Ok, we understand when $\Pi_\mu^{(N)} : \mathcal{M}_2^{\otimes N} \to \mathcal{M}_2^{\otimes N}$ is CP.

$$\boxed{\text{When is } \Pi_\mu^{(N)} \text{ decomposable ?}}$$

<u>Remember</u>: A priori we have to check that

$$\left(id_2^{\otimes N} \otimes \Pi_\mu^{(N)}\right)(X_{AB}) \geq 0$$

$\forall\, X_{AB}$ PPT.

Observation 2: $\Pi_\mu^{(N)} : \mathcal{M}_2^{\otimes N} \longrightarrow \mathcal{M}_2^{\otimes N}$ is decomposable

if and only if

"Pauli-diagonal" PPT matrix

$$\left(id_2^{\otimes N} \otimes \Pi_\mu^{(N)}\right)\left(C_{\Pi_{\tilde{\mu}}^{(N)}}\right) \geq 0$$

$\forall\ \Pi_{\tilde{\mu}}$ Pauli-diagonal s.th.

$\Pi_{\tilde{\mu}}^{(N)}$ & $\vartheta_2^{\otimes N} \circ \Pi_{\tilde{\mu}}^{(N)}$ are both completely positive.

Write: $CP_N \cap coCP_N$ for this set of $\Pi_{\tilde{\mu}}^{(N)}$.

We saw that

$$CP_N := \left\{ \Pi_\mu^{(N)} \text{ completely positive} \right\}$$

$$= \left\{ \Pi_\mu^{(N)} : \mu \in (\mathbb{R}^4)^{\otimes N} \text{ with } D^{\otimes N}\mu \text{ entrywise positive} \right\}$$

$$\hookrightarrow D = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

$\rightsquigarrow$ Polyhedral cone !

$\Longrightarrow$ $coCP_N$ & $CP_N \cap coCP_N$ are polyhedral cones as well.

Conclusion: To check whether $\Pi_\mu^{(N)} : M_2^{\otimes N} \to M_2^{\otimes N}$ is decomposable, we have to check whether

$$\left(id_2^{\otimes N} \otimes \Pi_\mu^{(N)}\right)\left(C_{\Pi_{\tilde{\mu}}^{(N)}}\right) \geq 0$$

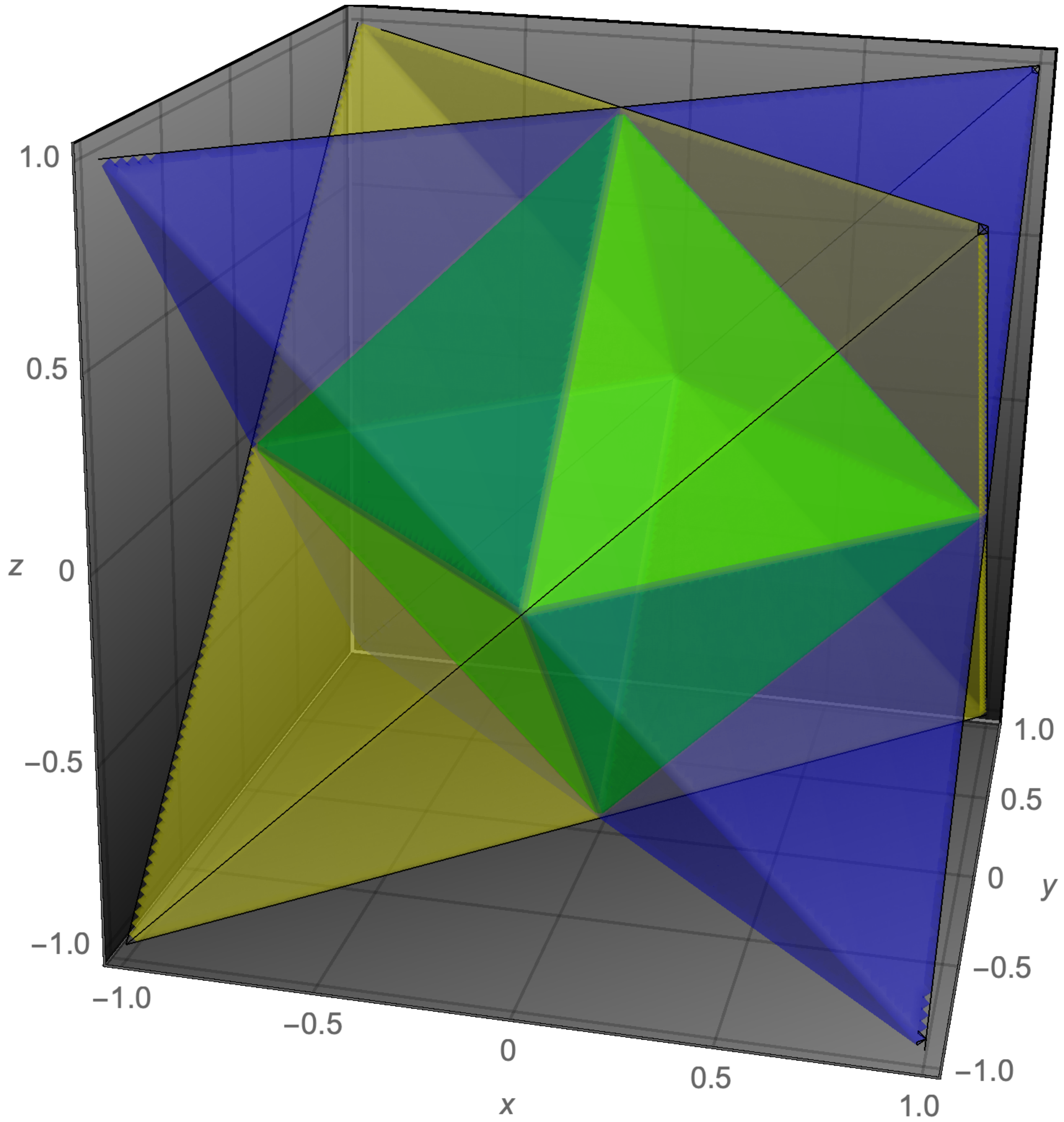
for the finite set of extremal rays $\Pi_{\tilde{\mu}}^{(N)}$ of the cone

$$CP_N \cap coCP_N .$$

$$\boxed{\text{How do the bases of these cones look like for } N=1\ ?}$$

$\rightsquigarrow$ Base of $CP_1 = \{ \underbrace{\Pi_\mu^{(1)} : M_2 \to M_2}_{\text{completely positive}} \}$

are the unital qubit channels.


1.0
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0
0.5
1.0
x
1.0
0.5
0
y
−0.5
−1.0

$CP_1 \cap coCP_1$ is the cone over the octahedron

$\rightsquigarrow$ 6 extremal rays !

For $N=2$ things are more complicated!

Thm (AMH '21)

The polyhedral cone $CP_2 \cap coCP_2$ has 252 extremal rays.

- 60 extremal rays of $\Pi_{\tilde{\mu}}^{(2)} : M_4 \to M_4$ for which

  $C_{\Pi_{\tilde{\mu}}^{(2)}}$ is separable.

- 192 extremal rays of $\Pi_{\tilde{\mu}}^{(2)} : M_4 \to M_4$ for which

  $C_{\Pi_{\tilde{\mu}}^{(2)}}$ is entangled.

These come in 3 orbits under a natural group action.

By checking the 252 inequalities

$$(id_2 \otimes id_2 \otimes \Pi_\lambda \otimes \Pi_\lambda)(C_{\tilde{\Pi}_\lambda}) \geq 0$$

for the extremal $\tilde{\Pi}_\lambda \in CP_2 \cap coCP_2$ we show

<u>Thm (AMH '21)</u>:

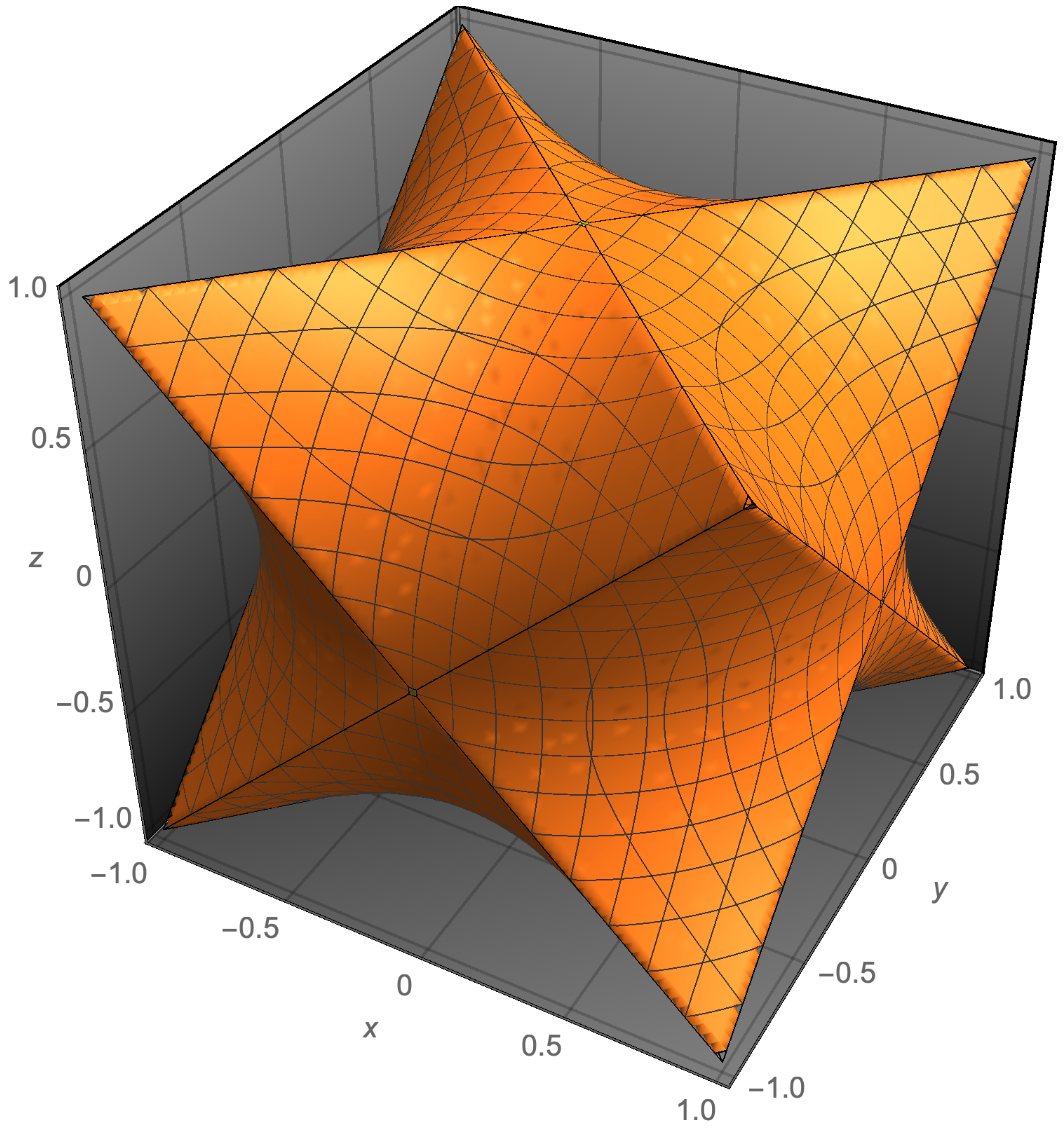
For Pauli-diagonal maps $\Pi_\lambda : \mathcal{M}_2 \to \mathcal{M}_2$ the following are equivalent:

(1) $\Pi_\lambda \otimes \Pi_\lambda$ positive

(2) $\Pi_\lambda \otimes \Pi_\lambda$ decomposable

(3) $$1 + \lambda_1^2 \geq \lambda_2^2 + \lambda_3^2$$
$$1 + \lambda_2^2 \geq \lambda_1^2 + \lambda_3^2$$
$$1 + \lambda_3^2 \geq \lambda_1^2 + \lambda_2^2 \, .$$

How does the set of all $\lambda \in \mathbb{R}^3$ look like for which $\Pi_\lambda \otimes \Pi_\lambda$ is positive/decomposable?



1.0
0.5
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−1.0
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0
x
0.5
1.0
−1.0
−0.5
0
y
0.5
1.0

Summary: We have seen how non-decomposable maps can arise as tensor products of decomposable maps.

$\rightsquigarrow$ e.g. $id_2 \otimes P$ for 2-positive P.

For qubits we have:

Thm: (AMH '21)

For $P: M_2 \to M_2$ the following are equivalent

(1) $P \otimes P$ positive.

(2) $P \otimes P$ decomposable

Higher tensor powers are interesting as well.

Thm (AMH '18):

If $P^{\otimes k}$ is decomposable $\forall k \in \mathbb{N}$, then P or $\vartheta \circ P$ completely positive.

$\rightsquigarrow$ Is there a map $P: M_{d_A} \to M_{d_B}$ s.th $P^{\otimes k}$ is positive $\forall k \in \mathbb{N}$ & neither P nor $\vartheta \circ P$ are completely positive?

$\rightsquigarrow$ Would give many non-decomposable maps!

Open questions:

- Are there maps $P: M_2 \to M_3$ s.th. $P \otimes P$ is positive but not decomposable?

- Are there maps $P: M_2 \to M_2$ s.th. $P \otimes P \otimes P$ is positive but not decomposable?

Thank you for your attention!

Thank you M.D Choi & A. Jamiotkowski for the nice results!

Open questions:

- Does this extend to 3rd tensor power:

  Are the following equivalent for $P: M_2 \to M_2$?

  (1) $P \otimes P \otimes P$ positive.

  (2) $P \otimes P \otimes P$ decomposable.

  $\rightsquigarrow$ Numerical evidence that this is the case.

- Characterize when general $P: M_d \to M_d$ satisfy

  $P^{\otimes k}$ positive for certain or all $k \in \mathbb{N}$.

THANK YOU FOR YOUR ATTENTION !