The Choi-Jamiolkowski Isomorphism: when Maths meets Physics

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Man-Duen Choi, "Completely positive linear maps on complex matrices". Linear Algebra and its Applications. 10, 285–290 (1975)

Andrzej Jamiołkowski, "Linear transformations which preserve trace and positive semidefiniteness of operators", Reports on Mathematical Physics 3, 275–278 (1972)

$$\dot{y} = y^{1/3}$$

$$y(0) = 0 \quad \text{(Cauchy)}$$

3 solutions:

$$y(t) = 0$$

$$y(t) = \pm\left(\frac{2}{3}t\right)^{3/2}$$

Expt: try it on the average physicist

Different expt (to try on the average physicist)

$$\ddot{x} = 6\,x^{1/3}$$

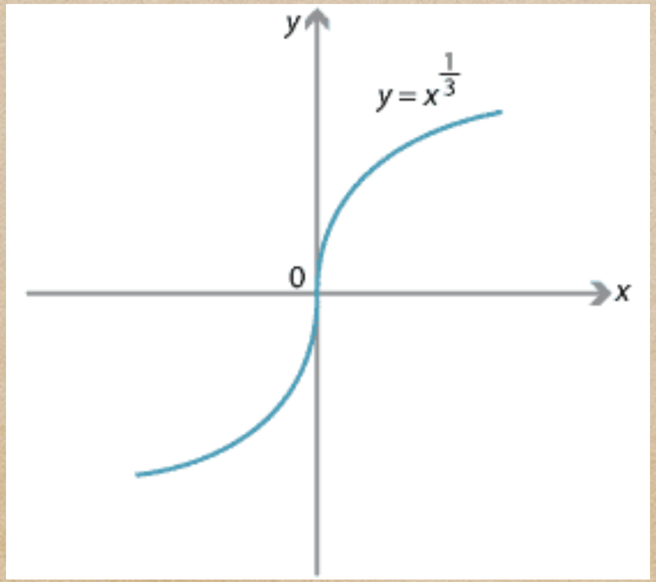
acceleration force

$$a = F/m$$

$$x(0) = 0, \dot{x}(0) = 0$$

2 solutions:
$$x(t) = 0$$
$$x(t) = t^3$$

$$F \sim x^{1/3}$$



potential ?

$$V \sim -x^{4/3}$$

when Maths meets Physics

- we beg for an explanation
- we implicitly demand that the solution (exists and) be unique
- therefore let's HOPE that $V \sim -x^{4/3}$ does not exist in Nature

when Maths meets Physics

- But let's also be a bit careful
- the solution "must" be unique... but
- not for problems involving boundary conditions
- not for hystereses
- not for phase transitions
- not for symmetry breaking phenomena
- ...
- for equations that describe physical time evolution

for equations that describe physical time evolution we want/expect that the solution be unique

(That's maybe why we don't easily accept quantum measurements, in which the system "decides" where to go)

in fact we want/expect even more (from a good physical evolution law like F=ma):
we want/expect to be able to choose ANY initial condition

In fact we want/expect even more (from a good physical law like F=ma):

we want/expect to be able to choose ANY initial condition

Otherwise we wouldn't call it a "law"

There is, in my humble opinion, a subtle interplay between a physical evolution law, expressed via a differential equation, and its Cauchy initial conditions. If some ICs are not possible/licit, we look for (demand?) an explanation

- OK, but what about Choi-Jamiolkowski?
- The answer is not simple
- Take an evolution law, but for an open quantum system

For example GKLS
(but a Q map, à la Kraus-Sudarshan, would be the same)

$$\dot{\rho} = L(\rho)$$

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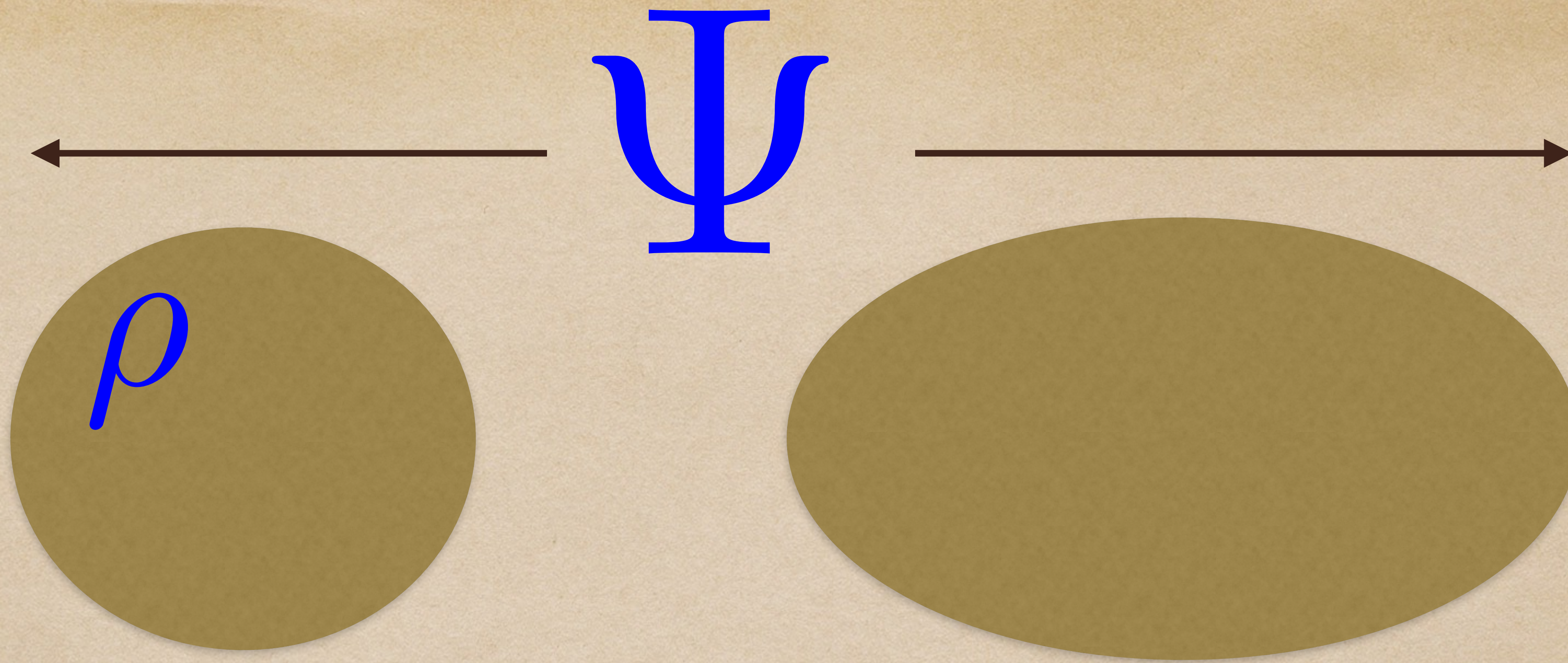
This is a (very) good physical law:
therefore we want/expect to be able to choose
ANY initial condition

But can we?

NO, unless L is CPTP

- But CP is a (big) problem
- We need a "construction"
- We need Choi-Jamiolkowski
- Otherwise we might not be able to freely choose any initial condition for $\dot{\rho} = L(\rho)$
- In some sense, CP and CJ enable us to freely choose any initial condition for what we deem to be a (good) physical law



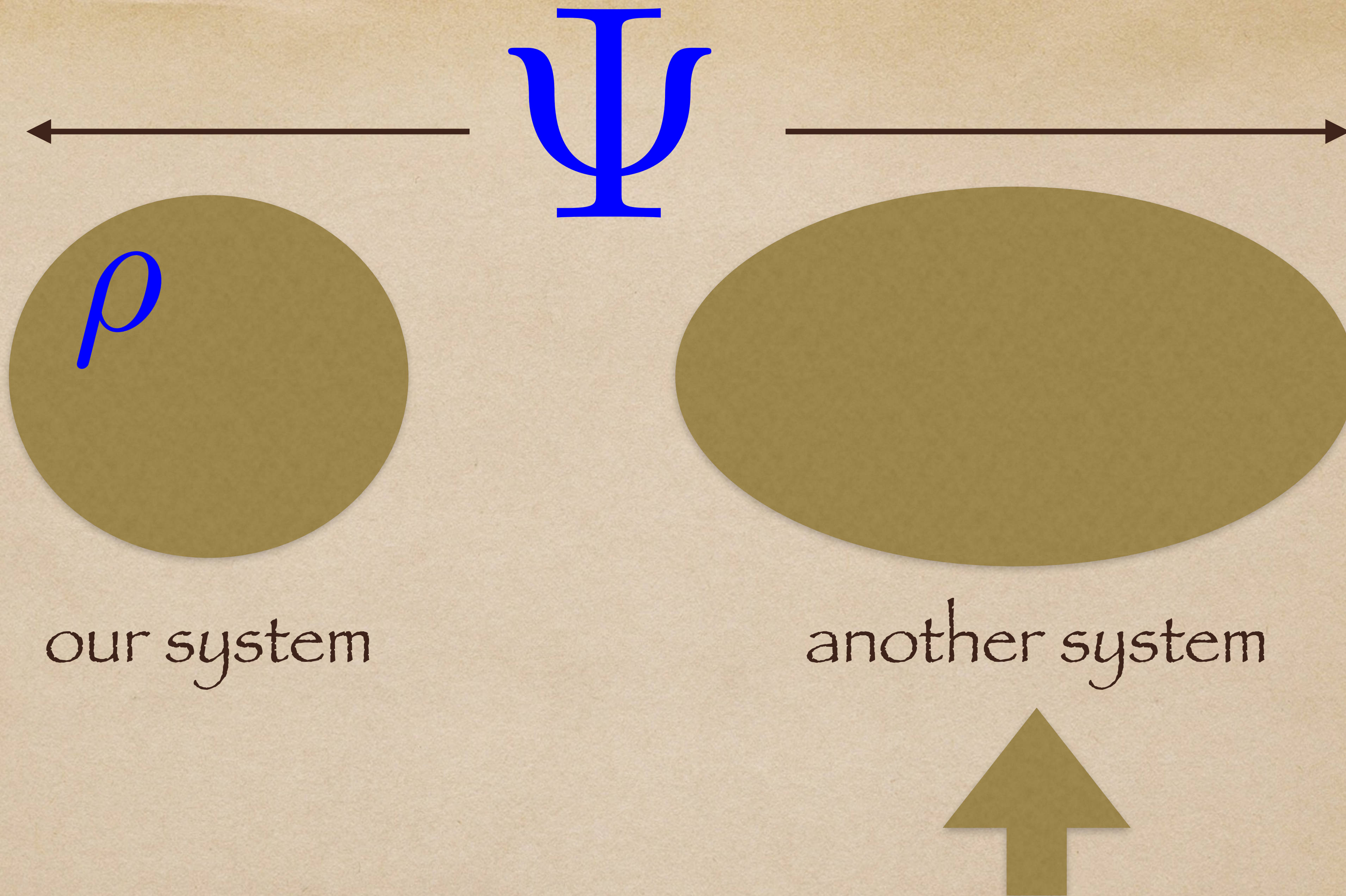
our system another system

$$\rho = \mathrm{Tr}_{\text{other system}}\Psi$$

There might be, unknown to us, (Q) correlations

No problem:

CP and CJ enable us to give any initial conditions for ρ



Well… in fact ANY other system

ANY? Careful: JC tell us that the other system must satisfy some requirements/characterization

$$\dot{\rho} = L(\rho)$$

- So in the end…
- We decide to constrain L
- And constrain the environment (or whatever)
- In order to be able to freely choose any initial condition

Reminder

- therefore let's HOPE that $V \sim -x^{4/3}$ does not exist in Nature
- And let's also hope that "wrong" L's in $\dot{\rho} = L(\rho)$ do not exist in Nature

when Maths meets Physics:
Always flabbergasting

Thank you