

Positive Maps and Entanglement in Real Hilbert Spaces

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What is unpleasant here, and indeed directly to be objected to, is the use of complex numbers. Ψ is surely fundamentally a real function.

Letter from Schrodinger to Lorentz. June 6th, 1926.

In *Quantum theory based on real numbers can be experimentally falsified*(Navascues et al), Dec. 27, 2021, a Bell-like experiment based on a network scenario is proposed that numerically detects if underlying state spaces are real or complex.

Trials of this experiment have now shown they are complex.

This motivated our work which looks for other similarities/differences between the real and complex case for various concepts like, entanglement and separability, positive maps, p-positive maps, the various characterizations of entanglement breaking maps and the PPT^2 conjecture.

In the real case many authors study these concepts for maps defined on the space of real symmetric matrices, but we prefer to look at maps on the full space of real matrices. Why?

Still have the Choi-Jamiolkowski isomorphism:

$$\Phi \in \mathcal{L}(M_d(\mathbb{R}), M_r(\mathbb{R})) \leftrightarrow C_\Phi = \sum_{i,j=1}^d E_{i,j} \otimes \Phi(E_{i,j}) \in M_d(\mathbb{R}) \otimes M_r(\mathbb{R}).$$

Separable and Entangled

Throughout \mathbb{K} stands for \mathbb{R} or \mathbb{C} . We set

$$\begin{aligned} SEP(\mathbb{K}^d, \mathbb{K}^r) &:= \left\{ \sum_I P_I \otimes Q_I : P_I \in PSD_d(\mathbb{K}), Q_I \in PSD_r(\mathbb{K}) \right\} \\ &\subseteq M_d(\mathbb{K}) \otimes M_r(\mathbb{K}). \end{aligned}$$

Any matrix in $(M_d(\mathbb{K}) \otimes M_r(\mathbb{K}))^+ \setminus SEP(\mathbb{K}^d, \mathbb{K}^r)$ is called \mathbb{K} -entangled.

If a real matrix is separable as a complex matrix, is it necessarily also separable as a real matrix? That is, is

$$SEP(\mathbb{R}^d, \mathbb{R}^r) \stackrel{?}{=} SEP(\mathbb{C}^d, \mathbb{C}^r) \cap (M_d(\mathbb{R}) \otimes M_r(\mathbb{R})) := CSEP(\mathbb{R}^d, \mathbb{R}^r).$$

NO!

$$SEP(\mathbb{R}^k, \mathbb{R}^n) \subsetneq CSEP(\mathbb{R}^k, \mathbb{R}^n)$$

Let $A = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ then

$$\begin{aligned} X = I_2 \otimes I_2 + A \otimes A &= \left(\begin{array}{cc|cc} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{array} \right) \\ &= 1/2(I_2 + A) \otimes (I_2 + A) + 1/2(I_2 - A) \otimes (I_2 - A), \end{aligned}$$

is \mathbb{C} -separable.

But note that if $Y = \sum_I P_I \otimes Q_I$ is \mathbb{R} -separable, then $id \otimes T(Y) = \sum_I P_I \otimes T(Q_I) = Y$. Since $id \otimes T(X) \neq X$, X is not \mathbb{R} -separable.

More generally we let

$$SEP_p(\mathbb{K}^d, \mathbb{K}^r) = \left\{ \sum_j |u_j\rangle\langle u_j| : u_j \in \mathbb{K}^d \otimes \mathbb{K}^r, \text{ rank at most } p \right\},$$

and

$$CSEP_p(\mathbb{R}^d, \mathbb{R}^r) = SEP_p(\mathbb{C}^d, \mathbb{C}^r) \cap (M_d(\mathbb{R}) \otimes M_r(\mathbb{R})).$$

Størmer's Theorem and Its Generalizations

Theorem: Let $\Phi : M_d(\mathbb{K}) \rightarrow M_r(\mathbb{K})$ be linear with $\Phi(X^*) = \Phi(X)^*$. Then Φ is p -positive if and only if

$$\text{Tr}(C_\Phi X) \geq 0, \forall X \in \text{SEP}_p(\mathbb{K}^d, \mathbb{K}^r).$$

In particular, if X is \mathbb{K} -entangled, then there is a positive map such that $\text{Tr}(C_\Phi X) < 0$.

Complexifications of Real Linear Maps

Note that every $X \in M_n(\mathbb{C})$ can be written uniquely as $X = A + iB$ with $A, B \in M_n(\mathbb{R})$. Given a real linear map $\Phi : M_d(\mathbb{R}) \rightarrow M_r(\mathbb{R})$ its *complexification* is the complex linear map $\tilde{\Phi} : M_d(\mathbb{C}) \rightarrow M_r(\mathbb{C})$ given by

$$\tilde{\Phi}(A + iB) = \Phi(A) + i\Phi(B)$$

and $C_\Phi = C_{\tilde{\Phi}}$

Theorem: Let $\Phi : M_d(\mathbb{R}) \rightarrow M_r(\mathbb{R})$ be linear. Then $\tilde{\Phi}$ is p -positive if and only if

$$\text{Tr}(C_\Phi X) \geq 0, \forall X \in \text{CSEP}_p(\mathbb{R}^d, \mathbb{R}^r).$$

In particular, if $X \in \text{CSEP}_p(\mathbb{R}^d, \mathbb{R}^r) \setminus \text{SEP}_p(\mathbb{R}^d, \mathbb{R}^r)$, then there is a p -positive map Φ such that $\tilde{\Phi}$ is not p -positive and $\text{Tr}(C_\Phi X) < 0$.

Entanglement Breaking Maps

Definition: A CP map $\Phi : M_d(\mathbb{K}) \rightarrow M_r(\mathbb{K})$ is called **\mathbb{K} -entanglement p-breaking** if for every n and every $P = (P_{i,j}) \in (M_n(\mathbb{K}) \otimes M_d(\mathbb{K}))^+$ we have that

$$id_n \otimes \Phi(P) = (\Phi(P_{i,j})) \in SEP_p(\mathbb{K}^n, \mathbb{K}^r).$$

Horodecki-Schor-Ruskai and Its Generalizations

Theorem: Let $\Phi : M_d(\mathbb{K}) \rightarrow M_r(\mathbb{K})$ be a linear map. Then the following are equivalent.

- (i) Φ is \mathbb{K} -entanglement p -breaking,
- (ii) $C_\Phi \in SEP_p(\mathbb{K}^d, \mathbb{K}^r)$,
- (iii) there exist matrices $(A_i)_{i=1}^k \subset M_{r,d}(\mathbb{K})$ such that $\text{rank}(A_i) \leq p$ for every i and $\Phi(X) = \sum_{i=1}^k A_i X A_i^*$,
- (iv) $\Phi = \Delta \circ \Gamma$ where $\Gamma : M_d(\mathbb{K}) \rightarrow I_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K})$ and $\Delta : I_k^\infty(\mathbb{K}) \otimes M_p(\mathbb{K}) \rightarrow M_r(\mathbb{K})$ are completely positive maps for some $k \geq 1$.
- (v) for every m and every p -positive map $\Psi : M_r(\mathbb{K}) \rightarrow M_m(\mathbb{K})$ the map $\Psi \circ \Phi$ is completely positive,
- (vi) for every n and every p -positive map $\Psi : M_n(\mathbb{K}) \rightarrow M_d(\mathbb{K})$ the map $\Phi \circ \Psi$ is completely positive.

Theorem: Let $\Phi : M_d(\mathbb{R}) \rightarrow M_r(\mathbb{R})$ be a CP map. Then the following are equivalent:

1. for every n and every $P = (P_{i,j}) \in (M_n(\mathbb{R}) \otimes M_d(\mathbb{R}))^+$ we have that

$$id_n \otimes \Phi(P) = (\Phi(P_{i,j})) \in CSEP_p(\mathbb{R}^n, \mathbb{R}^r).$$

2. $\tilde{\Phi}$ is \mathbb{C} -entanglement p -breaking,
3. $C_\Phi \in CSEP_p(\mathbb{R}^d, \mathbb{R}^r)$,
4. there exist matrices of rank at most p , $A_i \in M_{r,d}(\mathbb{C})$ such that $\Phi(X) = \sum_i A_i X A_i^*$,
5. there exists a finite dimensional real C^* -algebra $\mathcal{C} \subseteq \ell_k^\infty(\mathbb{C}) \otimes M_p(\mathbb{C})$ and CP maps $\Delta : M_d(\mathbb{R}) \rightarrow \mathcal{C}$, $\Psi : \mathcal{C} \rightarrow M_r(\mathbb{R})$ such that $\Phi = \Psi \circ \Delta$.

Here by “real C^* -algebra”, we mean a real vector subspace, closed under product and adjoint.

The PPT^2 Conjecture

A map $\Phi : M_d(\mathbb{K}) \rightarrow M_r(\mathbb{K})$ is called **PPT**(positive partial transpose) if Φ and $T \circ \Phi$ are both CP, where T denotes the transpose map.

PPT^2 Conjecture: If $\Phi : M_d(\mathbb{C}) \rightarrow M_d(\mathbb{C})$ is PPT, then $\Phi \circ \Phi$ is EB.

Here are a few things that we know:

- ▶ It is true for $d = 2, 3$ and for many families of maps.
- ▶ It is equivalent to the conjecture that if $\Phi_i : M_{d_i}(\mathbb{C}) \rightarrow M_{d_{i+1}}(\mathbb{C})$ are PPT, then $\Phi_2 \circ \Phi_1$ is EB.
- ▶ (Kennedy-Manor-P) If Φ is PPT, unital and idempotent, then its range is an abelian C^* -algebra in the Choi-Effros product, $\Phi(X) \star \Phi(Y) = \Phi(\Phi(X)\Phi(Y))$, and hence $\Phi \circ \Phi = \Phi$ is EB.
- ▶ (Kennedy-Manor-P) If Φ is PPT and either unital or trace-preserving, then $\lim_n d(\Phi^n, EB) = 0$.
- ▶ (Jaques-P-Rahaman) If Φ is PPT and both unital and trace-preserving, then there exists k such that Φ^k is EB.

The direct real analogue of this conjecture is false.

Define $\Phi : M_2(\mathbb{R}) \rightarrow M_2(\mathbb{R})$ with

$$C_\Phi = \frac{1}{2} \begin{bmatrix} I_2 & \gamma \\ -\gamma & I_2 \end{bmatrix} \quad \text{where} \quad \gamma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then Φ is PPT, unital, trace preserving and idempotent, $\Phi \circ \Phi = \Phi$, but $\Phi \circ \Phi$ is not \mathbb{R} -entanglement breaking. However, $\tilde{\Phi} \circ \tilde{\Phi}$ is \mathbb{C} -entanglement breaking.

The ITP^2 Conjecture

Seeking a reasonable real version.

A map $\Phi : M_d(\mathbb{R}) \rightarrow M_r(\mathbb{R})$ is **ITP** if it is CP and $T \circ \Phi = \Phi$.

ITP^2 Conjecture: If $\Phi : M_d(\mathbb{R}) \rightarrow M_d(\mathbb{R})$ is ITP, then $\Phi \circ \Phi$ is \mathbb{R} -EB.

Things we know:

- ▶ If complex PPT^2 true, then IPT^2 true.
- ▶ Do not know if they are equivalent.
- ▶ If Φ ITP, unital and idempotent, then the range of Φ in the Choi-Effros product is isomorphic to $\ell_k^\infty(\mathbb{R})$ for some k and $\Phi \circ \Phi$ is EB.
- ▶ If Φ ITP and either unital or trace-preserving, then $\lim_n d(\Phi^n, \mathbb{R} - EB) = 0$.
- ▶ If Φ ITP, both unital and trace-preserving, then there exists k such that Φ^k is \mathbb{R} -EB.

Thanks!