The time operator in quantum mechanics. A geometric approach to the problem of quantization.

Mansur Muzafarov

Faculty of Physics UW

29.11.2023

The arrival time problem in classical mechanics
Let's consider the motion of a free particle, characterized by the velocity v in the one-dimensional case. In this case, the motion of the particle is desribed by the following equation:

$$x(t) = x_0 + v \cdot t.$$

We ask the following question: what time does it take for a particle, whose initial position was x to pass through the point $x = 0$? The answer is very simple and follows immediately from the equation of motion of the particle:

$$t = -\frac{x}{v} = -\frac{mx}{p}.$$

The arrival time problem in quantum mechanics:
The quantum version of the problem consists in finding the quantum equivalent of the classical observable t:

$$t = -\frac{mx}{p}.$$

The naive answer:

$$\hat{t} = -m\frac{\hat{x}}{\hat{p}}$$

unfortunately has a lot of problems. Even after symmetrization:

$$\hat{t} = -\frac{1}{2}m\left(\hat{x}\frac{1}{\hat{p}} + \frac{1}{\hat{p}}\hat{x}\right)$$

doesn't define any self-adjoint operator.

Different approaches to solving the problem.

The arrival time problem in quantum mechanics:

All attempts to solve the time operator problem can be divided into two groups:

1) Direct attempts to find the structure of the time operator $\hat{t}$.

2) Attempts to find distribution of the arrival time operator.

Pauli's argument

According to classical mechanics, the evolution in time of any dinamic (autonomous) variable is desribed by the following equation:

$$\frac{dF}{dt} = \{F, H\}.$$

Let's consider dinamic variable T such that:

$$\frac{dT}{dt} = \{T, H\} = 1.$$

Pauli's argument

According to the canonical quantization rule, which is based on replacing Poisson brackets with commutators, above mentioned classical obsewable T corresponds to self-adjoint operator

$$[\hat{T}, \hat{H}] = i\hbar.$$

However, Wolfgang Pauli proved that if such an operator $\hat{T}$ existed then the hamiltonian of the system $\hat{H}$ would have to have an unbounded and continuous spectrum.

Geometric quantization

It turns out that using the basic tools of differential geometry helps to solve the arrival time operator problem.
Let's consider an observable which is linear with respect to momenta:

$$f(\mathrm{x}, \mathrm{p}) = X^k(\mathrm{x}) p_k,$$

where X - a certain vector field.
We know that the vector field X generates local one-parametric group of diffeomorphisms G_t^X.

The above diffeomorphism can be used for local transport of wave functions (which are not functions, but half-density):

$$\psi_{tr}(x) := (G^X_{-t})^* \psi(x).$$

It turns out that the above transformation is unitary if and only if the measurable mapping G^X_t is global.
Therefore, let's assume the globality of the measurable mapping group G^X_t. Let's introduce the following notation:

$$\hat{U}_t(\psi) := (G^X_{-t})^* \psi,$$

where $\hat{U}_t$ - the group of the unitary transformations.

The group of unitary transformations has the following form:

$$\hat{U}_t = \exp\left(\frac{it}{\hbar}\hat{f}\right),$$

where $\hat{f}$ - Hermitian operator, which in this case acts as a group generator.
Therefore, we can define the Hermitian operator $\hat{f}$, corresponding to the classical observable f in the following way:

$$\hat{f} := -i\hbar\mathcal{L}_X = -i\hbar\left(\frac{d}{dt}\hat{U}_t\right)_{t=0}$$

The time operator in the geometric approach

It's easy to see that we also can operate with vector fields, which are defined in the momentum representation. Indeed, according to canonical transformation (x, p) to $(p, -x)$, we can easily quantize observables linear with respect to the variable x:

$$f(x, p) = -X(p)x.$$

The problem of the time operator comes down to quantizing the vector field of the following form:

$$X = \frac{m}{p}\frac{\partial}{\partial p}$$

The corresponding local mapping group G_t^X unfortunately is not global:

$$G_t^X(p) = \begin{cases} \sqrt{p^2 + 2mt}, & p > 0 \\ -\sqrt{p^2 + 2mt}, & p < 0, \end{cases}$$

Idea

The idea of J. Kijowski (supported strongly by prof. R. Ingarden, who decided to publish it in ROMP) was that instead of consider the original obsevable T

$$T = -\frac{mx}{p}$$

we have to consider the following observable:

$$T_a := -\frac{mx}{|p|} = \mathrm{sgn}(p)\,T.$$

Hence, we differently treat the 'left-movers' and 'right-movers'. Let's find such variable s, that the field

$$X = \frac{m}{|p|}\frac{\partial}{\partial p} = \frac{\partial}{\partial s}$$

The solution of the problem

It is easy to see that this is the following variable:

$$s = \mathrm{sgn}(p) \cdot \frac{p^2}{2m}.$$

The corresponding group of diffeomorphisms now has the form:

$$G_t^X(s) = s + t,$$

which of course is a global mapping, and so there exists a self-adjoint operator $\hat{T}_a = -i\hbar\frac{\partial}{\partial s}$, corresponding to T_a.

Arrival time distribution

For wave packet, which is composed of 'right-movers' exclusively, both $\hat{T}_a$ and $\hat{T}$ coinside. In general case, there is an uncertainty principle between left and right movers.
The probability distribution of arrival time has the following structure:

$$\rho(T_a) \equiv \rho(T) = \left| \frac{1}{\sqrt{2\pi\hbar}} \int \exp\left(-\frac{i}{\hbar} Ts\right) \psi(s) ds \right|^2 .$$

The above result can also be written in the momentum representation.

First example

Let's find the arrival time distribution for a gaussian wave packet with an average starting position of $< x >= 10$ and an average momentum of $< p >= -5$ ('left-mover').

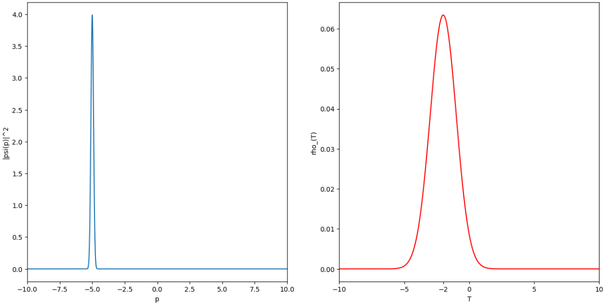

Figure: A Gaussian packet of $< x > = 10$ and $< p >= -5$.

Second example

As a final example, let's consider a gaussian wave packet with the same an average starting position of $< x >= 10$, and an average momentum of $< p >= -1$. But in this case, uncertainty of momenta $\Delta p = 2$. So, in this example we consider superposition of left and right movers.

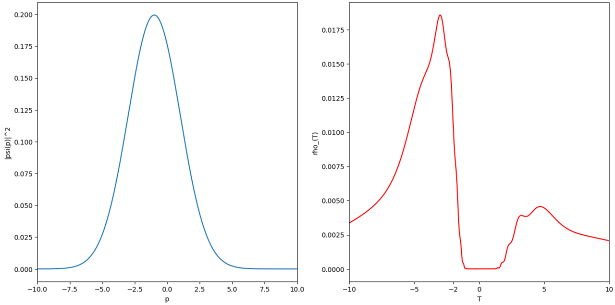

Figure: A Gaussian packet of $< x > = 10$ and $< p >=$ -1, $\Delta p = 2$.

The end

Thank you for attention!