Extremal jumps of circuit complexity of unitary evolutions generated by random Hamiltonians

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- State complexity: fix an initial state $|0\rangle$ and let $C_\varepsilon(|\psi\rangle)$ be the smallest k such that for some $V_1, \ldots, V_k \in \mathcal{G}$ we have:

$$d_{tr}(|\psi\rangle, V_1 \cdot V_2 \cdot \dots \cdot V_k |0\rangle) < \varepsilon$$

(trace distance)

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Note: lower bounding complexity for *specific* unitaries seems hopelessly difficult! Classical circuit lower bounds – $P \neq NP$ type of questions.

Random Quantum Circuits

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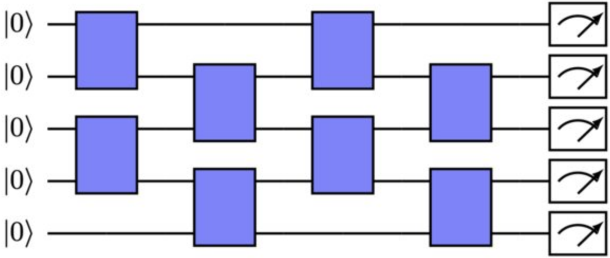

- choose local gates $U_{i,i+1}$ to be Haar random unitaries from 2-qubit unitaries $U(4)$, run for depth t.
- approaches Haar measure on $U(d)$ as $t \to \infty$; k-design properties known (Brandao, Harrow, Horodecki 2016).

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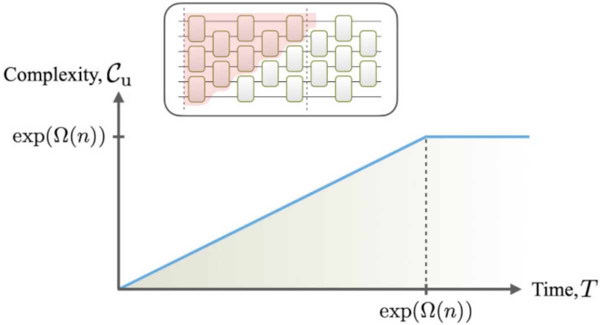

- **Brown-Susskind conjecture:** complexity of a random quantum circuit typically grows linearly with depth t until exponential value (no shortcuts). Proved in some settings (exact complexity) by Haferkamp, Faist, Kothakonda, Eisert, Yunger Halpern (2022) and Haferkamp (2023). *Still open!*

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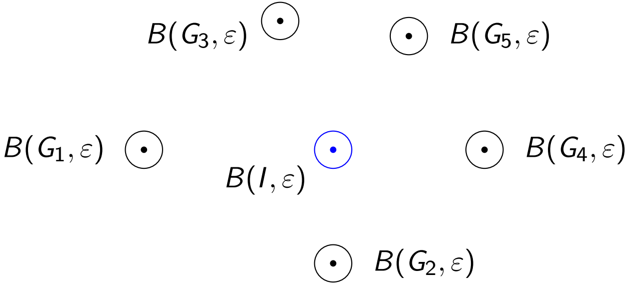
Haar random U: typically no structure = high complexity.
If complexity $> k$, then U lies outside the union of ε-balls around $\{G_1, G_2, \dots\} \in \mathcal{G}^k$ (words of length at most k)...



Unitary complexity

A single ball has volume $\sim \varepsilon^{d^2}$, so total volume $\leq |\mathcal{G}|^k \varepsilon^{d^2}$ – very small if $k \lesssim \frac{1}{\log|\mathcal{G}|} \cdot d^2 \log\left(\frac{1}{\varepsilon}\right)$

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Solovay-Kitaev theorem: every U has complexity at most $c(\mathcal{G}) \cdot d^2 \log^\gamma \left(\frac{1}{\varepsilon}\right)$ for some $\gamma < 4$. So *Haar random unitaries have typically maximal complexity*.

Our results

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Would like to prove something for local Hamiltonian with random couplings:

$$H = \sum_i g_i H_i$$

where H_i - local, $\|H_i\| = 1$, constants g_i – random couplings.

Difficult :(

Our results

Instead, consisder a "mean-field" random matrix GUE model (prominent e.g. in quantum chaos):

- The random Hamiltonian H is taken to be the $d \times d$ GUE matrix (Gaussian Unitary Ensemble, standard in random matrix theory, strongly interacting) – independent Gaussian entries:

$$H_{ii} \sim \text{real Gaussian of variance } 1$$
$$H_{ij} \sim \text{complex Gaussian of variance } 1$$
$$H_{ij} = \overline{H_{ji}}$$

- caveat: H is nonlocal

Our results

Theorem (Complexity jump for GUE evolutions)

Let $t_{esc} = C\varepsilon$, $t_{jump} = C'\varepsilon$ ($C' > C$). With high probability over $H \sim GUE(d)$

(a) *For all times time $t \in [0, t_{esc}]$ complexity of evolution U_t is trivial: $C_\varepsilon(U_t) = 0$.*

(b) *For any fixed time $t > t_{jump}$ complexity satisfies $C_\varepsilon(U_t) \geq \frac{C''\varepsilon^2}{\log|\mathcal{G}|} d^2$ (almost maximal wrt. d)*

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Much more powerful than non-rigorous heuristics based on frame potentials and moment computations (Cotler, Hunter-Jones, Liu, Yoshida 2017).

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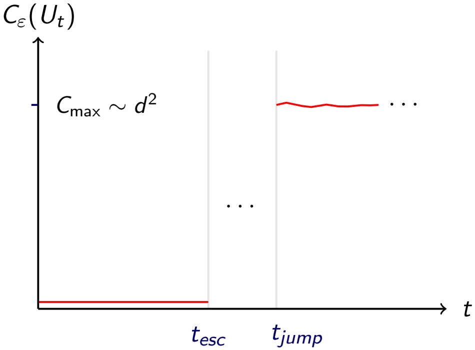
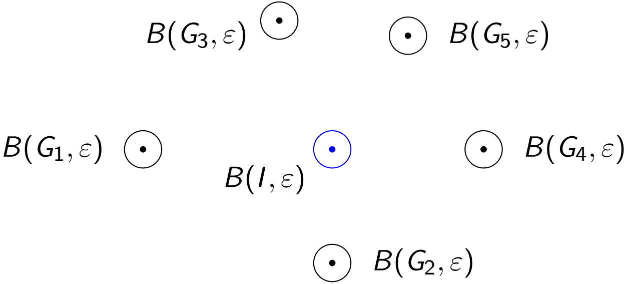


Very high level proof ideas

- Idea 1: GUE ensemble is unitarily invariant: the eigenbasis of H is Haar random! Use it to exclude all balls around $G \neq I$ (technically: concentration of measure on the unitary group)



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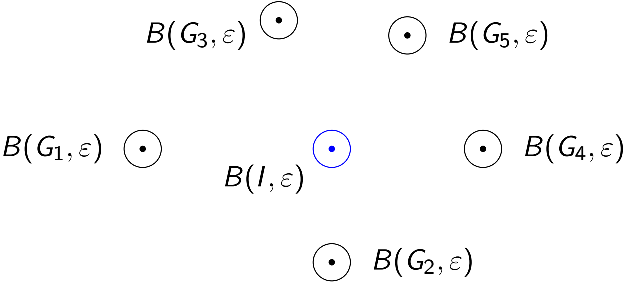

- Idea 2: use concentratino of measure for the **spectrum** of GUE matrix to control how spread out the distribution is (not too concentrated on the ball around identity).

Open problems and loose ends

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Thank you!